A Brüsszelátor-modell módosított változatainak vizsgálata

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Introduction, the studied models

Brusselator-model (Prigogine & Lefever [5])

- A, B, D, E: chemical reactants and products
- X: activator, Y: inhibitor
- assumption: A, B have constant concentration

$$
\begin{aligned}
A &\rightarrow X \\
2X+Y &\rightarrow 3X \\
B+X &\rightarrow Y+D \\
X &\rightarrow E
\end{aligned}
$$

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- assumption: A, B have constant concentration

$$\begin{aligned} A &\rightarrow X \\ 2X+Y &\rightarrow 3X \\ B+X &\rightarrow Y+D \\ X &\rightarrow E \end{aligned} \quad \rightsquigarrow \quad \left.\begin{aligned} \dot{X} &= f_1^0(X,Y) := A-(B+1)X+X^2Y \\ \dot{Y} &= f_2^0(X,Y) := BX-X^2Y \end{aligned}\right\}$$

Brusselator-model and modification

- A, B, D, E: chemical reactants and products
- X: activator, Y: inhibitor
- assumption: A, B have constant concentration
- $\sigma_1 > 0$, $\sigma_2 > 0$: strength of the feedback
- $\tau \geq 0$: time delay

$$\begin{array}{rcl} A & \to & X \\ 2X+Y & \to & 3X \\ B+X & \to & Y+D \\ X & \to & E \end{array} \quad \rightsquigarrow \quad \left.\begin{array}{rcl} \dot{X} & = & f_1^0(X,Y) := A-(B+1)X+X^2Y \\ \dot{Y} & = & f_2^0(X,Y) := BX - X^2Y \end{array}\right\}$$

Alfifi ([1]) $\rightsquigarrow$

$$\left.\begin{array}{rcl} \dot{X} & = & f_1^0(X,Y) + \sigma_1 X(\cdot - \tau) \\ \dot{Y} & = & f_2^0(X,Y) + \sigma_2 Y(\cdot - \tau) \end{array}\right\}$$

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$\tau = 0$ $\rightsquigarrow$

$$\left.\begin{array}{rcl} \dot{X} & = & f_1(X,Y) := A-(B+1-\sigma_1)X+X^2Y \\ \dot{Y} & = & f_2(X,Y) := BX + \sigma_2 Y - X^2Y \end{array}\right\}$$

MODELS WITH DIFFUSION

- $\Omega \subset \mathbb{R}^n$ bounded, connected spatial domain with piecewise smooth boundary $\partial\Omega$
- $d_1 > 0$, $d_2 > 0$: diffusion rates
- $X(r,t) > 0$, $Y(r,t) > 0$: concentrations at space $r \in \Omega$, time $t \geq 0$
- boundary condition: $(\mathbf{n} \cdot \nabla_r)\mathbf{S}(r,t) = 0$ for $(r,t) \in \partial\Omega \times \mathbb{R}_0^+$, ($\mathbf{n}$: outer unit normal to $\partial\Omega$)

$$\tau > 0 \quad \rightsquigarrow \quad \left.\begin{aligned} \partial_t X(r,t) &= f_1^0(X(r,t),Y(r,t)) + \sigma_1 X(r,t-\tau) + d_1\Delta_r X(r,t) \\ \partial_t Y(r,t) &= f_2^0(X(r,t),Y(r,t)) + \sigma_2 Y(r,t-\tau) + d_2\Delta_r X(r,t) \end{aligned}\right\} \qquad (t \geq 0,\ r \in \overline{\Omega})$$

$$\tau = 0 \quad \rightsquigarrow \quad \left.\begin{aligned} \partial_t X(r,t) &= f_1(X(r,t),Y(r,t)) + d_1\Delta_r X(r,t) \\ \partial_t Y(r,t) &= f_2(X(r,t),Y(r,t)) + d_2\Delta_r X(r,t) \end{aligned}\right\} \qquad (t \geq 0,\ r \in \overline{\Omega})$$

The studied models

- $\mathbf{S} := (X, Y)$, $\mathbf{f}^0 := (f_1^0, f_2^0)$, $\mathbf{f} := (f_1, f_2)$, $\mathbf{\Delta}_r \mathbf{S} := (\Delta_r X, \Delta_r Y)$,
- $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2) \in \mathbb{R}^{2\times 2}$, $D := \mathrm{diag}(d_1, d_2) \in \mathbb{R}^{2\times 2}$,

ODE:	$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{S})$
RD:	$\partial_t \mathbf{S} = \mathbf{f}(\mathbf{S}) + D \cdot \mathbf{\Delta}_r \mathbf{S}$
DDE:	$\dot{\mathbf{S}} = \mathbf{f}^0(\mathbf{S}) + \Sigma \cdot \mathbf{S}(\cdot - \tau)$
DDERD:	$\partial_t \mathbf{S} = \mathbf{f}^0(\mathbf{S}) + \Sigma \cdot \mathbf{S}(\cdot - \tau) + D \cdot \mathbf{\Delta}_r \mathbf{S}$

$$\mathbf{f}^0(X, Y) = \begin{bmatrix} A - (B+1)X + X^2 Y \\ BX - X^2 Y \end{bmatrix} \qquad \mathbf{f}(X, Y) = \begin{bmatrix} A - (B+1-\sigma_1)X + X^2 Y \\ BX + \sigma_2 Y - X^2 Y \end{bmatrix}$$

EQUILIBRIUM

$$\mathbf{f}(X,Y) = \mathbf{0} := (0,0) \iff \left.\begin{aligned} h_1(X) &= h_2(X) \\ Y &= h_1(X) \end{aligned}\right\}$$

$$\left.\begin{aligned} h_1(X) &:= \frac{(B+1-\sigma_1)X - A}{X^2} \\ h_2(X) &:= \frac{BX}{X^2 - \sigma_2} \end{aligned}\right\} \quad (X > 0)$$

Proposition 1 (cf. György, Kovács [2])

$$\exists!\, X^*, Y^* > 0 : \left.\begin{aligned} f_1(X^*,Y^*) &= 0 \\ f_2(X^*,Y^*) &= 0 \end{aligned}\right\} \iff \sigma_1 < 1$$

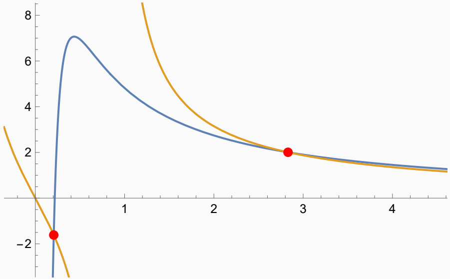


Figure 1: Example for two intersection points. $A = 1.340$, $B = 5.180$, $\sigma_1 = 0.026$, $\sigma_2 = 0.706$.

Stability and bifurcations without delay

ODE

$$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{S})$$

RD

$$\partial_t \mathbf{S} = \mathbf{f}(\mathbf{S}) + D \cdot \boldsymbol{\Delta}_r \mathbf{S} \qquad (\Omega \times \mathbb{R}_0^+)$$

$$(\mathbf{n} \cdot \nabla_r)\mathbf{S} = 0 \qquad (\partial\Omega \times \mathbb{R}_0^+)$$

$$\mathbf{f}(X, Y) = \begin{bmatrix} f_1(X, Y) \\ f_2(X, Y) \end{bmatrix} = \begin{bmatrix} A - (B + 1 - \sigma_1)X + X^2 Y \\ BX + \sigma_2 Y - X^2 Y \end{bmatrix}$$

Linearization at $\mathbf{S}^* := (X^*, Y^*)$

ODE

$$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{S}) \quad \rightsquigarrow \quad \dot{\mathbf{Z}} = \mathfrak{A}\mathbf{Z}$$

RD

$$\left.\begin{aligned} &\partial_t \mathbf{S} = \mathbf{f}(\mathbf{S}) + D \cdot \boldsymbol{\Delta}_r \mathbf{S} \quad (\Omega \times \mathbb{R}_0^+) \\ &(\mathbf{n} \cdot \nabla_r)\mathbf{Z} = 0 \quad (\partial\Omega \times \mathbb{R}_0^+) \end{aligned}\right\} \quad \rightsquigarrow \quad \left.\begin{aligned} &\dot{\mathbf{Z}} = \mathfrak{A}\mathbf{Z} + D \cdot \boldsymbol{\Delta}_r \mathbf{Z} \quad (\Omega \times \mathbb{R}_0^+) \\ &(\mathbf{n} \cdot \nabla_r)\mathbf{Z} = 0 \quad (\partial\Omega \times \mathbb{R}_0^+) \end{aligned}\right\}$$

Linearization at $S^* := (X^*, Y^*)$

ODE

$$\mathfrak{A} := J(\mathbf{f})(\mathbf{S}^*)$$

RD

eigenfunction expansion (cf. Kovács [4]):

$$\mathbf{\Psi}(\mathbf{r},t) = \sum_{n=0}^{\infty} \psi_n(\mathbf{r}) \exp\left(\mathfrak{A}_n t\right) \mathbf{\Psi}_{0_n} \qquad ((\mathbf{r},t) \in \overline{\Omega} \times \mathbb{R}_0^+),$$

where

$$\mathfrak{A}_n := \mathfrak{A} - \lambda_n D \qquad \mathbf{\Psi}_{0_n} := \int_\Omega \mathbf{Z}_0\left(\mathbf{r}\right) \psi_n(\mathbf{r})\, \mathrm{d}\mathbf{r}$$

$$\Delta\psi = -\lambda\psi, \qquad \left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0.$$

Linearization at $S^* := (X^*, Y^*)$

ODE

$$\mathfrak{A} := J(\mathbf{f})(\mathbf{S}^*)$$

$$\boxed{\mathbf{\Delta}(z) := z^2 - \mathrm{Tr}(\mathfrak{A})z + \det(\mathfrak{A})} \qquad (z \in \mathbb{C})$$

RD

$$\mathfrak{A}_n := \mathfrak{A} - \lambda_n D$$

$$\boxed{\mathbf{\Delta}_n(z) := z^2 - \mathrm{Tr}(\mathfrak{A}_n)z + \det(\mathfrak{A}_n)} \qquad (z \in \mathbb{C},\ n \in \mathbb{N}_0)$$

Characteristic polynomials

$K := 2X^*Y^* - (1 + B - \sigma_1)$

ODE

$$\mathbf{\Delta}(z) := p^0 + p^1 z + z^2 \qquad (z \in \mathbb{C})$$

$$p^0 := [X^*]^2(1 - \sigma_1) + \sigma_2 K,$$
$$p^1 := [X^*]^2 - \sigma_2 - K$$

RD

$$\mathbf{\Delta}_n(z) := p_n^0 + p_n^1 z + z^2 \qquad (z \in \mathbb{C})$$

$$p_n^0 := p^0 + \lambda_n(d_1([X^*]^2 - \sigma_2) - d_2 K) + d_1 d_2 \lambda_n^2,$$
$$p_n^1 := p^1 + \lambda_n(d_1 + d_2)$$

Stability of $\Delta(z) = z^2 + p^1 z + p^0$

Assume that $\sigma_1, \sigma_2 < 1$.

Lemma 2

$$p^0 = [X^*]^2(1-\sigma_1) + \sigma_2 K > 0$$

Proposition 3

$$(X^*, Y^*) \text{ is stable} \qquad \Longleftrightarrow \qquad Q(X^*) > 0,$$

where

$$Q(x) := (\sigma_2 - 2(1-\sigma_1))x^2 + 2Ax + \sigma_2(1 + B - \sigma_1 - \sigma_2) \qquad (x \in \mathbb{R}).$$

$$(X^*, Y^*) \text{ is stable} \quad \overbrace{\Longleftrightarrow}^{\text{Routh-Hurwitz cr.}} \quad p^0 > 0 \wedge;\ p^1 > 0$$

$$\Delta(z) = z^2 + \underbrace{([X^*]^2 - \sigma_2 - K)}_{=Q(X^*)} z + \underbrace{[X^*]^2(1-\sigma_1) + \sigma_2 K}_{>0}$$

Let denote by $x_\pm$ the roots of Q:

$$x_\pm := \frac{-A \pm \sqrt{A^2 - \sigma_2(\sigma_2 - 2(1-\sigma_1))(1+B-\sigma_1-\sigma_2)}}{\sigma_2 - 2(1-\sigma_1)}$$

Proposition 4

$Q(X^*) > 0 \iff$ *one of the below three cases is true:*

- $\sigma_2 - 2(1-\sigma_1) = 0$ *and*
  - $B \geq 1 - \sigma_1$ *or*
  - $B < 1 - \sigma_1$ *and* $X^* > \dfrac{(1-\sigma_1)(1-\sigma_1-B)}{A}$
- $\sigma_2 - 2(1-\sigma_1) > 0$ *and*
  - $B + 1 \geq \sigma_1 + \sigma_2$ *or*
  - $B + 1 < \sigma_1 + \sigma_2$ *and* $X^* > x_+$
- $\sigma_2 - 2(1-\sigma_1) < 0$ *and*
  - $B + 1 \geq \sigma_1 + \sigma_2$ *and* $X^* < x_-$ *or*
  - $B + 1 < \sigma_1 + \sigma_2$ *and* $A^2 - \sigma_2(\sigma_2 - 2(1-\sigma_1))(1+B-\sigma_1-\sigma_2) > 0$ *and* $x_+ < X^* < x_-$

Hopf bifurcation

Theorem 5 (cf. György, Kovács [2])

Assumption: $\sigma_2 > -4A^2(B-(1-\sigma_1))/(B+1-\sigma_1)^3$.

- $2\sigma_1 + 3\sigma_2 \neq 2 \Longrightarrow$ *Hopf bifurcation occur at* A_+^*;
- $2\sigma_1 + 3\sigma_2 \neq 2$ *and* $\sigma_1 + \sigma_2 > 1 + B \Longrightarrow$ *Hopf bifurcation occur at* $A_\pm^*$;

where

$$A_\pm^* := -\frac{\sigma_2(1+B-(\sigma_1+\sigma_2)) + \left[X_\pm^*\right]^2(-2+2\sigma_1+\sigma_2)}{2X_\pm^*}$$

$$X_\pm^* := \sqrt{\frac{-1+B+\sigma_1+2\sigma_2 \pm \sqrt{(-1+B+\sigma_1)^2+8\sigma_2 B}}{2}},$$

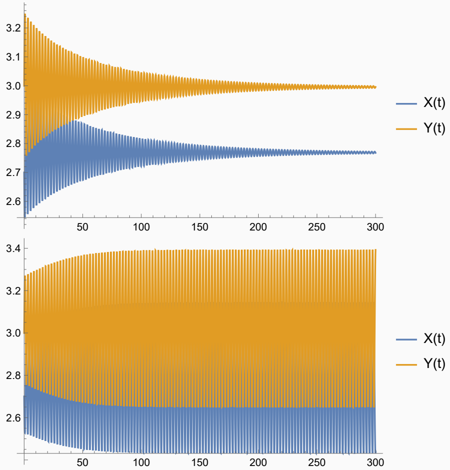


Figure 2: Solutions of the ODE system with parameter values $B = 8.23$, $\sigma_1 = 0.22$, $\sigma_2 = 0.06$ and $A = 1.97$ (top image) resp. $A = 1.98$ (bottom image) and initial conditions $X_0 = 2.70$, $Y_0 = 3.0$.

STABILITY OF $\Delta_n(z) = z^2 + p_n^1 z + p_n^0$

Assumptions: $p^1 > 0$, $K > 0$. **Question:** d_1, d_2 can destabilize?

$$\forall n \in \mathbb{N}: \quad p_n^1 = p^1 + \lambda_n(d_1 + d_2) > 0 \quad \Longrightarrow \quad \text{no stable} \to \text{unstable via Hopf}$$

$$p_n^0 = d_1\lambda_n \underbrace{\left([X^*]^2 - \sigma_2 + d_2\lambda_n\right)}_{>0} + p^0 - Kd_2\lambda_n$$

$$\Downarrow$$

$$p_n^0 > 0 \quad \Longleftrightarrow \quad d_1 > d_{1,n}^* := \frac{Kd_2\lambda_n - p^0}{\lambda_n([X^*]^2 - \sigma_2 + d_2\lambda_n)} \qquad (n \in \mathbb{N})$$

Proposition 6

stability of S^* is
- *changed* $\Longleftrightarrow$ $\exists\, n \in \mathbb{N}: \; d_1 \leq d_{1,n}^*$,
- *preserved* $\Longleftrightarrow$ $d_1 > d_1^* := \max\left\{d_{1,n}^* \mid n \in \mathbb{N}_0\right\}$

EXAMPLE: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$

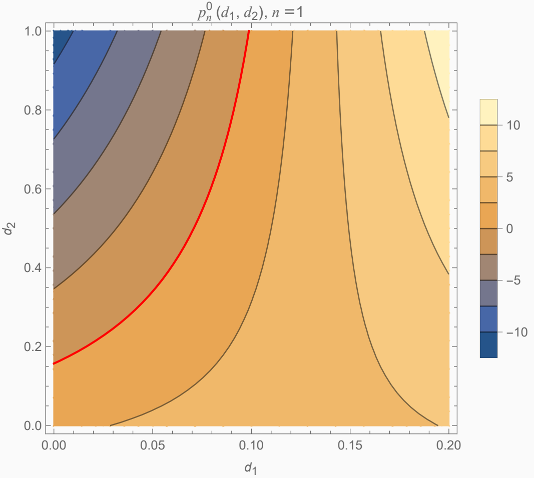


Figure 3: Contour plot of $(d_1, d_2) \mapsto p_n^0$ when $n = 1$.

Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$

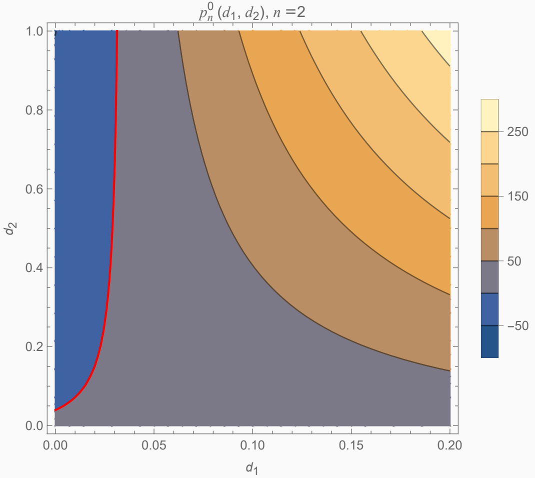


Figure 3: Contour plot of $(d_1, d_2) \mapsto p_n^0$ when $n = 2$.

Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$

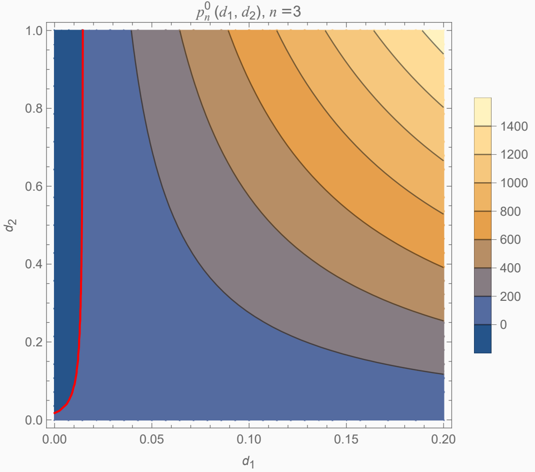


Figure 3: Contour plot of $(d_1, d_2) \mapsto p_n^0$ when $n = 3$.

Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$

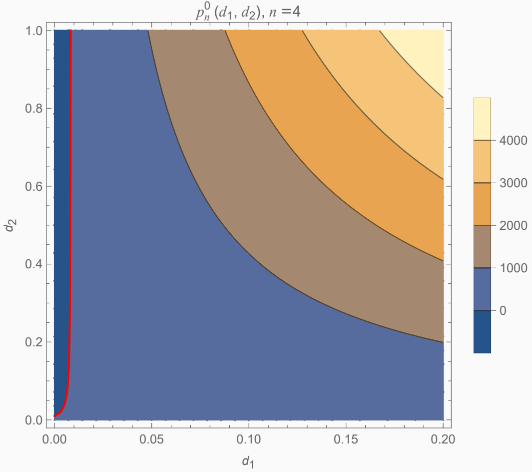


Figure 3: Contour plot of $(d_1, d_2) \mapsto p_n^0$ when $n = 4$.

Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$

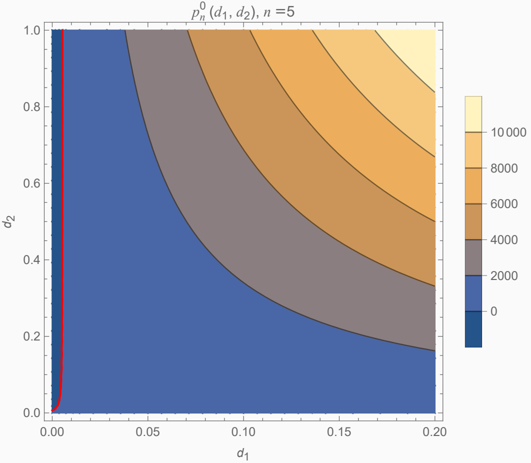
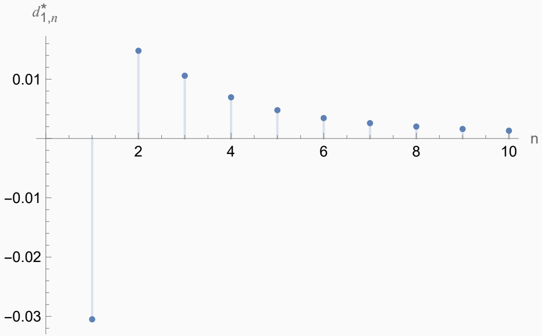


Figure 3: Contour plot of $(d_1, d_2) \mapsto p_n^0$ when $n = 5$.

EXAMPLE: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$, $d_2 = 0.1$

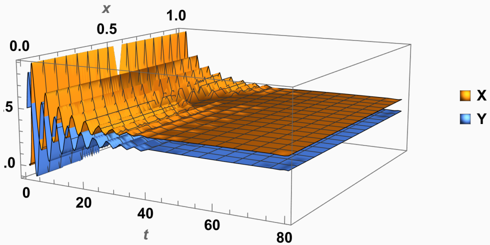


Conjecture: $\exists n_0 \in \mathbb{N} \quad : \quad d^*_{1,n} \downarrow \; (n > n_0)$

Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0,1)$, $d_2 = 0.1$

$$d_1^* \approx 0.015$$

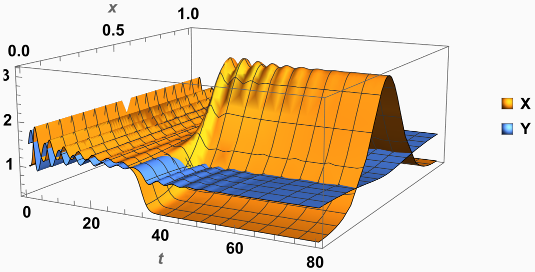
Solutions with $d_1 = 0.1 > d_1^*$



Example: $A = 0.6$, $B = 1.4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\Omega = (0, 1)$, $d_2 = 0.1$

$$d_1^* \approx 0.015$$

Solutions with $d_1 = 0.01 < d_1^*$



Stability and bifurcations with delay

INTRODUCING TIME DELAY

DDE

$$\dot{\mathbf{S}} = \mathbf{f^0}(\mathbf{S}) + \Sigma \cdot \mathbf{S}(\cdot - \tau)$$

DDE+RD

$$\partial_t \mathbf{S} = \mathbf{f^0}(\mathbf{S}) + \Sigma \cdot \mathbf{S}(\cdot - \tau) + D \cdot \mathbf{\Delta}_r \mathbf{S} \quad (\Omega \times \mathbb{R}_0^+)$$

$$(\mathbf{n} \cdot \nabla_r)\mathbf{S} = 0 \quad (\partial\Omega \times \mathbb{R}_0^+)$$

$$\mathbf{f}^0(X,Y) = \begin{bmatrix} f_1^0(X,Y) \\ f_2^0(X,Y) \end{bmatrix} = \begin{bmatrix} A - (B+1)X + X^2Y \\ BX - X^2Y \end{bmatrix}$$

Linearization at $S^* := (X^*, Y^*)$

DDE

$$\mathfrak{A} := J(\mathbf{f}^0)(\mathbf{S}^*) \quad \mathfrak{B} := \Sigma$$

$$\mathbf{\Delta}_0(z;\tau) := \det\left(zI_2 - \mathfrak{A} - \mathfrak{B}e^{-z\tau}\right) \qquad (z \in \mathbb{C},\ \tau > 0)$$

DDE+RD

$$\mathfrak{A} := J(\mathbf{f}^0)(\mathbf{S}^*) \quad \mathfrak{B} := \Sigma$$

$$\mathbf{\Delta}_n(z;\tau) := \det\left(zI_2 - \mathfrak{A}_n - \mathfrak{B}e^{-z\tau}\right) \qquad (z \in \mathbb{C},\ \tau > 0)$$

Characteristic functions

DDE

$$\mathbf{\Delta}_0(z;\tau) := p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \qquad (z \in \mathbb{C},\ \tau \geq 0)$$

$$\begin{aligned} p(z) &:= [X^*]^2 + ([X^*]^2 - K + \sigma_1)z + z^2 \\ q(z) &:= -[X^*]^2\sigma_1 + \sigma_2(K - \sigma_1) - (\sigma_1 + \sigma_2)z \\ r(z) &:= \sigma_1\sigma_2 \end{aligned}$$

DDE+RD

$$\mathbf{\Delta}_n(z;\tau) := p_n(z) + q_n(z)e^{-z\tau} + r_n(z)e^{-2z\tau} \qquad (z \in \mathbb{C},\ \tau \geq 0)$$

$$\begin{aligned} p_n(z) &:= [X^*]^2 + \lambda_n(d_1[X^*]^2 + d_2(-K + \sigma_1)) + d_1 d_2 \lambda_n^2 \\ &\quad + ([X^*]^2 - K + \sigma_1 + (d_1 + d_2)\lambda_n)z + z^2 \\ q_n(z) &:= -\sigma_1[X^*]^2 + \sigma_2(K - \sigma_1) - (d_1\sigma_2 + d_2\sigma_1)\lambda_n - (\sigma_1 + \sigma_2)z \\ r_n(z) &:= \sigma_1\sigma_2 \end{aligned}$$

Assumption:

(X^*, Y^*) is asymptotically stable when $\tau = 0$

Question:

(X^*, Y^*) remains stable for all $\tau > 0$

OR

becomes unstable at some $\tau > 0$?

Assumption:

$$\left.\begin{aligned}\boldsymbol{\Delta}_0(z,0) = p(z) + q(z) + r(z) = p^0 + p^1 z + z^2 \\ \boldsymbol{\Delta}_n(z,0) = p_n(z) + q_n(z) + r_n(z) = p_n^0 + p_n^1 z + z^2\end{aligned}\right\} \rightsquigarrow \text{Hurwitz stable}$$

Question:

DDE

$\forall\, \tau > 0 :\quad \boldsymbol{\Delta}_0(z,\tau)$ is stable,

OR

$\exists\, \tau > 0 :\quad \boldsymbol{\Delta}_0(z,\tau)$ is unstable?

DDE+RD

$\forall\, \tau > 0 :\quad \forall\, n \in \mathbb{N}_0\ \boldsymbol{\Delta}_n(z,\tau)$ is stable,

OR

$\exists\, \tau > 0 :\quad \exists\, n \in \mathbb{N}\ \boldsymbol{\Delta}_n(z,\tau)$ is unstable?

A GENERAL RESULT ON THE ESTIMATION OF THE STABILITY INTERVAL

$$\boldsymbol{\Delta}(z;\tau) = p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \qquad (z \in \mathbb{C},\ \tau \geq 0)$$

$$p(z) = z^2 + a_1 z + a_0, \quad q(z) = b_1 z + b_0, \quad r(z) = c \qquad (z \in \mathbb{C})$$

where $a_1, a_0, b_1, b_0, c \in \mathbb{R}$

Assumption:

$\boldsymbol{\Delta}(z;0)$ is Hurwitz stable

Question:

$\bar{\tau} =?$ such that $\boldsymbol{\Delta}(z;\tau)$ is Hurwitz stable if $\tau < \bar{\tau}$

Theorem 7 (Kovács, György, Gyúró [3])

$$a_0 + b_0 + c > 0 \ \wedge \ a_1 + b_1 \qquad \Longrightarrow \qquad \boldsymbol{\Delta}(\cdot;\tau) \quad \textit{is stable, if} \quad \tau < \frac{a_1 - |b_1|}{|b_0| + 2|c|}$$

Stability interval for the DDE case

Coefficients:

$$a_0 = (X^*)^2$$

$$a_1 = (X^*)^2 - K + \sigma_1$$

$$b_0 = -\sigma_1([X^*]^2 + \sigma_2) + \sigma_2 K$$

$$b_1 = -(\sigma_1 + \sigma_2)$$

$$c = \sigma_1 \sigma_2$$

Assumption:

$$(X^*)^2 - \sigma_2 - K > 0$$

$$(1 - \sigma_1)(X^*)^2 + \sigma_2 K > 0$$

Theorem 7 $\Longrightarrow$

$$\tau < \overline{\tau_0} := \frac{[X^*]^2 - \sigma_2 - K}{|-\sigma_1([X^*]^2 + \sigma_2) + \sigma_2 K| + 2\sigma_1\sigma_2} \quad \Longrightarrow \quad \boldsymbol{\Delta}_0(z;\tau) \text{ is Hurwitz stable}$$

Stability interval for the DDE+RD case

Coefficients:

$$a_0^n = a_0 + (d_1[X^*]^2 - d_2(K - \sigma_1))\lambda_n + d_1 d_2 \lambda_n^2$$

$$a_1^n = a_1 + (d_1 + d_2)\lambda_n$$

$$b_0^n = b_0 - (d_1\sigma_2 + d_2\sigma_1)\lambda_n$$

$$b_1^n = b_1$$

$$c^n = c$$

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$$b_1^n = b_1$$

$$c^n = c$$

Assumption: $\frac{d_1}{d_2} \geq \frac{[X^*]^2 - \sigma_2}{K}$ and $\boldsymbol{\Delta}_0(z;\tau)$ is Hurwitz stable

$\Downarrow$

$$a_1^n + b_1^n = \underbrace{a_1 + b_1}_{>0} + \underbrace{(d_1 + d_2)\lambda_n}_{\geq 0} > 0$$

$$a_0^n + b_0^n + c = \underbrace{a_0 + b_0 + c}_{>0} + \underbrace{(d_1([X^*]^2 - \sigma_2) - d_2 K)\lambda_n}_{\geq 0} + \underbrace{d_1 d_2 \lambda_n^2}_{\geq 0} > 0$$

Stability interval for the DDE+RD case

Coefficients:

$$a_0^n = a_0 + (d_1[X^*]^2 - d_2(K - \sigma_1))\lambda_n + d_1 d_2 \lambda_n^2$$

$$a_1^n = a_1 + (d_1 + d_2)\lambda_n$$

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$$b_1^n = b_1$$

$$c^n = c$$

Theorem 7 $\implies$

$$\boxed{\overline{\tau_n} := \frac{[X^*]^2 - \sigma_2 - K + (d_1 + d_2)\lambda_n}{|-\sigma_1([X^*]^2 + \sigma_2) + \sigma_2 K - (d_1\sigma_2 + d_2\sigma_1)\lambda_n| + 2\sigma_1\sigma_2}}$$

Proposition 8

$\tau < \overline{\tau_n} \implies$ $\mathbf{\Delta}_n$ *is Hurwitz stable*

$\tau < \overline{\tau} := \inf\{\overline{\tau_n} \mid n \in \mathbb{N}\} \implies$ (X^*, Y^*) *is Hurwitz stable*

ANALYSIS OF $\overline{\tau}$

Let: $L := [X^*]^2 - \sigma_2 - K$, $M := \sigma_1([X^*]^2 - \sigma_2) - \sigma_2 K + 2\sigma_1\sigma_2$

Assumption: $\sigma_1([X^*]^2 + \sigma_2) - \sigma_2 K > 0$. Then:

$$\overline{\tau_n} = \frac{L + (d_1 + d_2)\lambda_n}{M + (d_1\sigma_2 + d_2\sigma_1)\lambda_n} = \frac{d_1 + d_2}{\sigma_1 d_2 + \sigma_2 d_1} \cdot \left(1 + \frac{\frac{\sigma_1 d_2 + \sigma_2 d_1}{d_1 + d_2} \cdot L - M}{M + (d_1\sigma_2 + d_2\sigma_1)\lambda_n} \right)$$

$$\frac{\sigma_1 d_2 + \sigma_2 d_1}{d_1 + d_2} \cdot L - M < 0 \quad \Longrightarrow \quad \overline{\tau} = \overline{\tau_0} = \frac{L}{M}$$

$$\frac{\sigma_1 d_2 + \sigma_2 d_1}{d_1 + d_2} \cdot L - M \geq 0 \quad \Longrightarrow \quad \overline{\tau} = \lim_{n \to +\infty} \overline{\tau_n} = \frac{d_1 + d_2}{\sigma_1 d_2 + \sigma_2 d_1}$$

Proposition 9

(X^*, Y^*) *is Hurwitz stable, if* $\tau < \overline{\tau} = \min\left\{ \frac{L}{M}, \frac{d_1 + d_2}{\sigma_1 d_2 + \sigma_2 d_1} \right\}$

A GENERAL RESULT ON THE POSSIBILITY OF STABILITY CHANGE

Let p, q, r be polynomials, consider the function

$$\boldsymbol{\Delta}(z;\tau) = p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \qquad (z \in \mathbb{C},\ \tau \geq 0).$$

Based on Wu, Ren [6] and Kovács, György, Gyúró [3]:

Proposition 10

Assume that $\boldsymbol{\Delta}(z;0)$ is Hurwitz stable. Then

$$\exists\, \tau > 0 \ :\ \boldsymbol{\Delta}(z;\tau) \text{ is unstable} \quad \Longleftrightarrow \quad \exists\, \omega \in \mathbb{R}\setminus\{0\} \ :\ R(\omega) = 0 \wedge (F(\omega) > 0 \wedge G(\omega) > 0),$$

$$\begin{aligned}
R(\omega) &= \{q_I(\omega)(r_I(\omega) - p_I(\omega)) - q_R(\omega)(p_R(\omega) - r_R(\omega))\}^2 \\
&\quad + \{q_R(\omega)(p_I(\omega) + r_I(\omega)) - q_I(\omega)(p_R(\omega) + r_R(\omega))\}^2 \\
&\quad - \left\{p_R^2(\omega) + p_I^2(\omega) - r_R^2(\omega) - r_I^2(\omega)\right\}^2, \\
F(\omega) &= (r_R(\omega) + p_R(\omega))^2 + (r_I(\omega) - p_I(\omega))^2 - q_R^2(\omega), \\
G(\omega) &= (r_I(\omega) + p_I(\omega))^2 + (r_R(\omega) - p_R(\omega))^2 - q_I^2(\omega),
\end{aligned}$$

$$\begin{aligned}
p(\imath\omega) &=: p_R(\omega) + \imath p_I(\omega), \\
q(\imath\omega) &=: q_R(\omega) + \imath q_I(\omega), \\
r(\imath\omega) &=: r_R(\omega) + \imath r_I(\omega).
\end{aligned}$$

A frequently used special case

$$\mathbf{\Delta}(z;\tau) = p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau} \qquad (z \in \mathbb{C},\ \tau \geq 0)$$

$$p(z) = z^2 + a_1 z + a_0, \quad q(z) = b_1 z + b_0, \quad r(z) = c \qquad (z \in \mathbb{C})$$

where $a_1, a_0, b_1, b_0, c \in \mathbb{R}$

$$\Downarrow$$

$$R(\omega) + F(\omega) \cdot G(\omega) = (b_0 b_1 - 2a_1 c)^2 \omega^2 \qquad (\omega \in \mathbb{R})$$

$$\Downarrow$$

$$b_0 b_1 - 2a_1 c \neq 0 \ \wedge\ \exists\, \omega^* \neq 0 :\ R(\omega^*) = 0 \quad \Longrightarrow \quad \operatorname{sgn}(F(\omega^*)) = \operatorname{sgn}(G(\omega^*)).$$

Proposition 11

Assume that $\mathbf{\Delta}(z;0)$ is Hurwitz stable. Then

$$\exists\, \tau > 0 \ :\ \mathbf{\Delta}(z;\tau) \text{ is unstable}$$

$$\Updownarrow$$

$$\exists\, \omega \in \mathbb{R} \setminus \{0\} \ :\ R(\omega) = 0 \ \wedge\ (F(\omega) > 0 \vee G(\omega) > 0)$$

BACK TO Δ_0 AND Δ_n

$$n = 0 \quad : \quad G_0(\omega) = \omega^2((1 + B + [X^*]^2)^2 - (\sigma_1 + \sigma_2)^2) + (\ldots)^2 > 0$$

$$n > 0 \quad : \quad G_n(\omega) = G_0(\omega) + (\ldots)^2 > 0$$

$$\Downarrow$$

DDE

(X^*, Y^*) becomes unstable, if

$$\exists \omega \in \mathbb{R} \setminus \{0\} \; : \; R_0(\omega) = 0$$

DDE+RD

(X^*, Y^*) becomes unstable, if

$$\exists n \in \mathbb{N}_0, \, \exists \omega \in \mathbb{R} \setminus \{0\} \; : \; R_n(\omega) = 0$$

R CORRESPONDING TO Δ_0

$$R_0(\omega) := R_0^0 + R_0^2\omega^2 + R_0^4\omega^4 + R_0^6\omega^6 + \omega^8 \qquad (\omega \in \mathbb{R})$$

$$R_0^0 := ([X^*]^4 - \sigma_1^2\sigma_2^2)^2 - A^2_{\sigma_1,\,\sigma_2}([X^*]^2 - \sigma_1\sigma_2)^2,$$

$$\begin{aligned} R_0^2 := \; & -2A_{\sigma_2,\,\sigma_1} \cdot A_{\sigma_1,\,\sigma_2}([X^*]^2 - \sigma_1\sigma_2) \\ & +2(A^2_{1,\,1} - 2\,[X^*]^2)([X^*]^4 - \sigma_1^2\sigma_2^2) \\ & -(A_{1,\,1} \cdot A_{\sigma_1,\,\sigma_2} - (\sigma_1+\sigma_2)([X^*]^2 + \sigma_1\sigma_2))^2 \end{aligned}$$

$$\begin{aligned} R_0^4 := \; & 2([X^*]^4 - \sigma_1^2\sigma_2^2) + (A^2_{1,\,1} - 2\,[X^*]^2)^2 - A^2_{\sigma_2,\,\sigma_1} \\ & +2(\sigma_1+\sigma_2)((\sigma_1+\sigma_2)([X^*]^2 + \sigma_1\sigma_2) - A_{1,\,1} \cdot A_{\sigma_1,\,\sigma_2}) \end{aligned}$$

$$R_0^6 := -4\,[X^*]^2 + 2A^2_{1,\,1} - (\sigma_1+\sigma_2)^2,$$

$$A_{k,l} := k \cdot [X^*]^2 + l \cdot (1 + B - 2X^*Y^*)$$

Analysis of the coefficients (ongoing work)

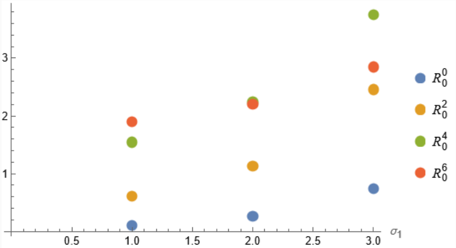
Lemma 12

Assume that $\boldsymbol{\Delta}_0(z;0)$ *is Hurwitz stable. Then*

$$R_0^0 > 0.$$

Conjectures from numerical experiments:

- Sign of the coefficients:
  - $(R_0^0,)\ R_0^4 > 0$ and $R_0^2,\ R_0^6 < 0$ or
  - $(R_0^0,)\ R_0^4,\ R_0^2,\ R_0^6 > 0$
- $\min\{|R_0^0|,\ |R_0^2|,\ |R_0^4|,\ |R_0^6|\} = |R_0^6|$, if the values of the parameters are large enough



$A = 0.7,\ \sigma_2 = 0.3,$
$B \in \{j \cdot 0.1\ :\ j \in \{1, 20\}\},$
x-axis: $\sigma_1 \in \{0.1,\ 0.2,\ 0.3\}$

R CORRESPONDING TO Δ_n

$$R_n(\omega) := R_n^0 + R_n^2\omega^2 + R_n^4\omega^4 + R_n^6\omega^6 + \omega^8 \qquad (\omega \in \mathbb{R})$$

$$R_n^0 = d_1^4 d_2^4 \lambda_n^8 + \mathcal{O}(\lambda_n^7)$$

$$R_n^2 = 2d_1^2 d_2^2 (d_1^2 + d_2^2)\lambda_n^6 + \mathcal{O}(\lambda_n^5)$$

$$R_n^4 = (d_1^4 + 4d_1^2 d_2^2 + d_2^4)\lambda_n^4 + \mathcal{O}(\lambda_n^3)$$

$$R_n^6 = 2(d_1^2 + d_2^2)\lambda_n^2 + \mathcal{O}(\lambda_n)$$

$$\Downarrow$$

for fixed A, B, σ_1, σ_2, d_1, d_2 :

$$\exists\, n_0 \in \mathbb{N} : \ \forall n \geq n_0 \ : \ R_n(\omega) > 0 \ \forall \omega \in \mathbb{R}$$

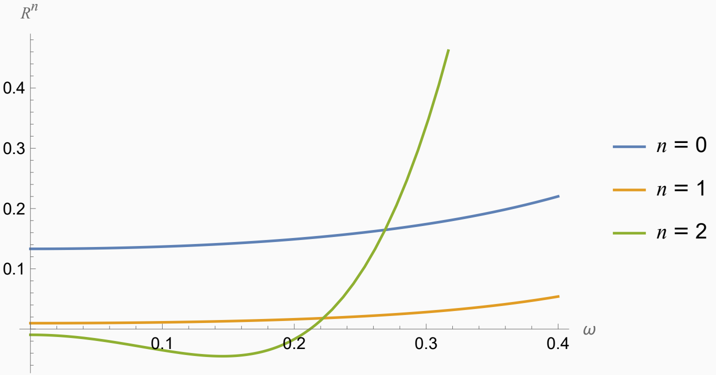


Figure 4: Example for R^n with parameter values $A = 0.3$, $B = 2.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $d_1 = 0.035$, $d_2 = 0.073$, $\Omega = (0, 1)$.

Hopf bifurcation

Calculation of τ^*:

$$R_n(\omega^*) = 0 \quad \rightsquigarrow \quad \boxed{\tau_j^* := \begin{cases} \dfrac{\arccos(C(\omega^*)) + j \cdot 2\pi}{\omega^*} & (\text{ if } S(\omega^*) \geq 0) \\[2ex] \dfrac{2\pi - \arccos(C(\omega^*)) + j \cdot 2\pi}{\omega^*} & (\text{ if } S(\omega^*) < 0) \end{cases} \quad j \in \mathbb{N}_0}$$

$$S(\omega) \quad := \quad \frac{q_I(\omega) \cdot \{r_R(\omega) + p_R(\omega)\} - q_R(\omega) \cdot \{r_I(\omega) + p_I(\omega)\}}{p_I^2(\omega) + p_R^2(\omega) - r_I^2(\omega) - r_R^2(\omega)}$$

$$C(\omega) \quad := \quad \frac{q_R(\omega) \cdot \{r_R(\omega) - p_R(\omega)\} + q_I(\omega) \cdot \{r_I(\omega) - p_I(\omega)\}}{p_I^2(\omega) + p_R^2(\omega) - r_I^2(\omega) - r_R^2(\omega)}$$

Theorem 13 (cf. Kovács, György, Gyúró [3])

Assumption: $\mathbf{\Delta}(\imath\omega^*; \tau^*) = 0$. *Hopf bifurcation occur at* $\tau = \tau^*$ *iff* $\operatorname{sgn}(\Im(\mathfrak{A})) = \pm 1$, *where*

$$\mathfrak{A} := \bar{q}q' + 2\bar{r}r' + (2\bar{r}q' + \bar{q}p')e^{\imath\omega^*\tau^*} + \bar{q}r'e^{-\imath\omega^*\tau^*} + 2\bar{r}p'e^{2\imath\omega^*\tau^*}$$

EXAMPLE: IN THE ABSENCE OF DIFFUSION

$A = 1.4$, $B = 7.5$, $\sigma_1 = 0.4$, $\sigma_2 = 0.08$

Equilibrium: $(X^*, Y^*) \approx (2.7,\ 2.8)$

Calculated critical values: $\omega^* \approx 3.04$, $\tau^* \approx 1.47$

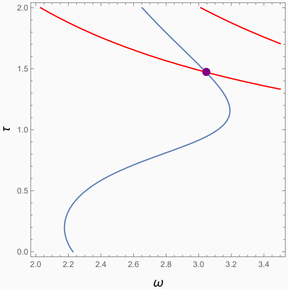


Figure 5: $\Re(\mathbf{\Delta}_0(\imath\omega;\tau)) = 0$ (blue line) és $\Im(\mathbf{\Delta}_0(\imath\omega;\tau)) = 0$ (red line).

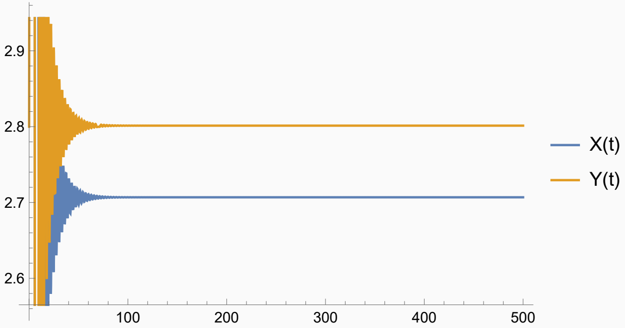


Figure 6: $\tau = 0$

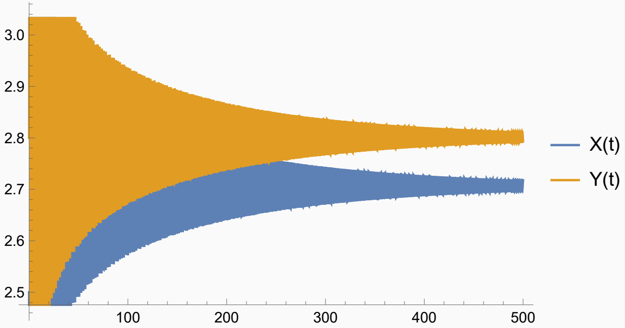


Figure 6: $\tau = 1.46$

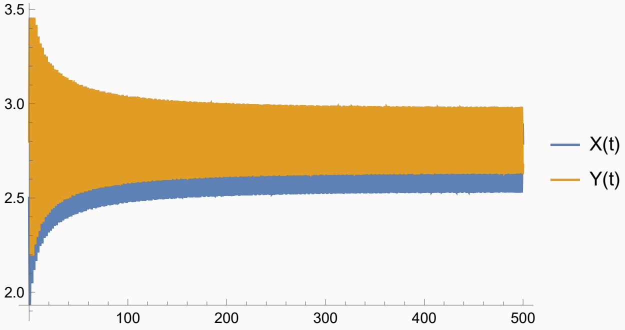


Figure 6: $\tau = 1.48$

Köszönöm szépen a figyelmüket!

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