Weighted maximal operators of Fejér means of Walsh-Fourier series

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The aim of this talk is to present a summary of the results published in the article:

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Fourier analysis on compact topological groups

A modern approach to the theory of Fourier series is the study of orthonormal systems defined on topological groups. A topological group G is group which is also a topological spaces, where the group operation and the inverse operation are continuous.

Characters: are continuous functions $\psi : G \to \mathbb{C}$ such that

$$\psi(xy) = \psi(x)\psi(y), \quad |\psi(x)| = 1 \qquad (x, y \in G).$$

If G is compact, then the set of all characters is discrete. Moreover, there is an unique non-negative regular measure μ on the Borel sets of G which is two-sided translation invariant and $\mu(G) = 1$, called Haar measure.

If G is an abelian group, then a system formed by all characters of G is an orthonormal and complete system in $L^2(G)$.

Fourier analysis on compact topological groups

Relevant examples are $\mathbb{T} := \{\exp(\imath x) \mid x \in [0, 2\pi)\}$ the multiplicative group of complex numbers in the unit circle, and the equivalent group $[0, 2\pi)$ with modulo 2π addition.

We define the topology on $\mathbb{T}$ as a subspace of the topology of $\mathbb{C}$. This induces a topology on $[0, 2\pi)$.

The Haar measure on $[0, 2\pi)$ is the Lebesgue measure divided by 2π. This induces the Haar measure on $\mathbb{T}$.

The set of all characters on $\mathbb{T}$:

$$\psi_n(\theta) = \theta^n \qquad (\theta \in \mathbb{T},\ n \in \mathbb{Z}).$$

The set of characters on $[0, 2\pi)$ (complex trigonometric system):

$$\psi_n(x) = \exp(\imath n x) \qquad (x \in [0, 2\pi),\ n \in \mathbb{Z}).$$

The dyadic group

Another example is $\mathbb{Z}_m$ the cyclic group of order m with discrete topology.

The Haar measure is the one that assigns to each singleton the measure $1/m$.

The set of characters on $\mathbb{Z}_m$:

$$\varphi_n(x) = \exp(2\pi\imath nx/m) \qquad (n \in \{0,\, 1, \dots,\, m-1\},\; x \in \mathbb{Z}_m).$$

Note that for $m = 2$

$$\varphi_n(x) = (-1)^{nx} \qquad (n \in \{0, 1\},\; x \in \mathbb{Z}_2),$$

that is

$$\varphi_0 \equiv 1 \qquad \text{and} \qquad \varphi_1(x) = \begin{cases} 1 & (x = 0) \\ -1 & (x = 1). \end{cases}$$

The dyadic group

The dyadic group: $G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_2$ is the complete product of $\mathbb{Z}_2$.

$x \in G \iff x = (x_0, x_1, \dots)$, where $x_k \in \{0, 1\}$.

$x + y = (x_0 + y_0 \mod 2,\ x_1 + y_1 \mod 2, \dots)$.

We assume the product topology and measure of $\mathbb{Z}_2$, and consider the characters φ_0 and φ_1.

The set of characters on G (Walsh-Paley system):

$$w_n(x) := \prod_{k=0}^{\infty} \varphi_{n_k}(x_k) = (-1)^{\sum_{k=0}^{\infty} n_k x_k} \qquad \big(x = (x_0, x_1, \dots) \in G\big),$$

and $(n_0, n_1, \dots)$ is the binary expansion of $n \in \mathbb{N}$, that is

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k \in \{0, 1\}.$$

The dyadic group

The dyadic intervals of G: For every $n \in \mathbb{N}$ denote

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\}, \quad I_0(x) := G, \quad I_n := I_n(0).$$

The set of dyadic intervals form a countable base of the topology. This is metrizable, indeed the map from G onto $[0, 1]$ defined by

$$|x| := |(x_0, x_1, x_2, \dots)| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}$$

is a norm.

A Walsh polynomial is a finite linear a finite linear combination of Walsh functions. We denote the collection of Walsh polynomials by $\mathcal{P}$, and it coincides with the set of the finite linear combination of characteristics functions of dyadic intervals.

The dyadic group

Remark. The original Walsh systems and and its classic rearrangements were originally defined on the interval $[0, 1)$. The fact that the Walsh functions can be viewed as characters of the dyadic group, was discovered independently by Fine and Vilenkin. The map

$$[0,1) \ni x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \to (x_0, x_1, x_2, \dots) \in G,$$

where $x_k \in \{0, 1\}$, connects the dyadic group with a structure on $[0, 1)$ formed by the dyadic addition, the topology generated by the dyadic intervals

$$I_k(i) := \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right) \quad (i = 0, \dots, 2^k - 1), \qquad I_k := I_k(0),$$

and the Lebesgue measure. However, the numbers $x = p/2^n$, called dyadic rationals, have two different expansions of which we only consider the one terminates in 0's. This leads to some inaccuracies, for instance the dyadic addition is not associative and the character property is not valid for Walsh functions defined on $[0, 1)$.

The Lebesgue spaces

The space L^0 is the set of functions which are a.e. limits of sequences in $\mathcal{P}$.

The space L^p is the set of $f \in L^0$ such that

$$\|f\|_p := \left(\int_G |f|^p \, d\mu \right)^{1/p} < \infty \quad (0 < p < \infty),$$

and

$$\|f\|_\infty := \inf\{y \in \mathbb{R} \colon |f(x)| \le y \text{ for a.e. } x \in G\} < \infty.$$

Notice that L^p is a Banach space for each $1 \le p \le \infty$ and $\mathcal{P}$ is dense in L^p for $1 \le p < \infty$. Moreover

$$L^q \subset L^p$$

if $0 < p < q \le \infty$.

On the other hand, $\| \cdot \|_p$ is a quasi-norm for $p < 1$.

The Lebesgue spaces

The space $L^{p,\infty}$ (or *weak* $- L^p$) is the set of $f \in L^0$ such that

$$\|f\|_{L^{p,\infty}} := \sup_{\lambda>0} y \cdot (\mu\{x \in G\colon |f| > \lambda\})^{1/p} < \infty \qquad (0 < p < \infty).$$

This "norm" is actually a quasi-norm. In addition

$$L^p \subset L^{p,\infty}.$$

Let $1 \leq p, q \leq \infty$. The T sublinear operator is said to be of type (p, q) if there is a constant $c > 0$ such that

$$\|Tf\|_q \leq c\|f\|_p \qquad (f \in L^p),$$

and it is said to be of weak type (p, q) if there is a constant $c > 0$ such that

$$\|Tf\|_{L^{p,\infty}} \leq c\|f\|_p \qquad (f \in L^p).$$

Martingal Hardy Spaces

In Fourier analysis the concept of Fourier coefficient

$$\widehat{f}_k := \langle f, w_k \rangle = \int_G f w_k \, d\mu \qquad (k \in \mathbb{N}),$$

plays a prominent role, but this requires that f be integrable. To generalize this concept we use martingal theory.

Denote by $\mathcal{A}$ the Borel sets of G, and

$$\mathcal{A}_n = \sigma\big(\{I_n(x) \colon x \in G\}\big) \qquad (n \in \mathbb{N}).$$

Then,

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A} \qquad \text{and} \qquad \mathcal{A} = \sigma\Big(\bigcup_{n=0}^{\infty} \mathcal{A}_n\Big).$$

Martingal Hardy Spaces

The conditional expectation operator of $g \in L^1$ relative to $\mathcal{A}_n$ is

$$E_n g(x) = 2^n \int_{I_n(x)} g(t)\, d\mu(t) \qquad (n \in \mathbb{N}).$$

A sequence $f = (f_n \colon n \in \mathbb{N})$ of integrable functions is said to be a dyadic martingale if

- f_n is $\mathcal{A}_n$ measurable for all $n \in \mathbb{N}$,
- $E_n f_m = f_n$ for all $m \geq n$.

Note that if $f \in L^1$, then $(E_n f \colon n \in \mathbb{N})$ is a dyadic martingale.

The space of L^p-bounded martingales is the set of all dyadic martingale such that

$$\|f\|_{L^p} := \sup_{n \in \mathbb{N}} \|f_n\|_p < \infty \quad (0 < p \leq \infty).$$

Martingal Hardy Spaces

The Fourier coefficients of the dyadic martingale f is defined by

$$\widehat{f}_k := \lim_{n\to\infty} \int_G f_n w_k d\mu.$$

The partial sums of a Walsh-Fourier series of the dyadic martingale f is defined by

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k w_k \qquad (n \in \mathbb{N}).$$

If $f \in L^1$, then $S_{2^n} f = E_n f$.

The Fejér means of Fourier series of the dyadic martingale f is defined by

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbb{N}^+).$$

Martingal Hardy Spaces

The maximal function of the dyadic martingale f is defined by

$$f^* := \sup_{n\in\mathbb{N}} |f_n|.$$

The martingale Hardy space H_p is the set of all dyadic martingale such that

$$\|f\|_{H_p} := \|f^*\|_p < \infty \quad (0 < p \leq \infty).$$

The spaces H_p and L^p are equivalents if $1 < p \leq \infty$.

Let X and Y be spaces with norm (or quasi-norm) $\|.\|_X$ and $\|.\|_Y$ respectively. We say that the operator $T : X \to Y$ is bounded from X to Y if there exists a $C > 0$ constant such that

$$\|Tf\|_Y \leq C\|f\|_X \qquad (f \in X).$$

Preliminary results

The maximal operator of Fejér means is

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|.$$

- σ^* of week type $(1,1)$ (Schipp).
- σ^* is bounded from H_1 to L_1 (Fujii and Simon).
- σ^* is bounded from H_p to L_p for $p > 1/2$ (Weisz).
- The above result is not true for $p \leq 1/2$ (Simon, Weisz, Goginava).
- There exists a martingale $f \in H_p$ $(p \leq 1/2)$, such that

$$\sup_{n\in\mathbb{N}} |\sigma_n f|_p = +\infty$$

  (Goginava).
- σ^* is bounded from $H_{1/2}$ to $L^{1/2,\infty}$ (Weisz).

Preliminary results

Weighted maximal operators of Fejér means were also studied. Tephnadze proved that the operator

$$\overset{\sim}{\sigma}^{*,p} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

is bounded from H_p to L^p where $p < 1/2$, and the rate of the sequence in the denominator can not be improved.

In case $p = 1/2$, Tephnadze also proved analogical results for

$$\overset{\sim}{\sigma}^{*} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\log^2 (n+1)}.$$

Main results

The aim of our paper was to improve $\overset{\sim}{\sigma}^{*,p}$ replacing the weights $(n+1)^{1/p-2}$ by more general, but "optimal" weights using the sequence

$$\rho(n) := \max\{k \in \mathbb{N} \colon n_k \neq 0\} - \min\{k \in \mathbb{N} \colon n_k \neq 0\}$$

and an arbitrary nonnegative and nondecreasing function $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$ which satisfies the condition

$$\sum_{n=1}^{\infty} 1/\varphi^p(n) < \infty.$$

Main results

Theorem

Let $0 < p < 1/2$, $f \in H_p$ *and* $\varphi : \mathbb{N}^+ \to \mathbb{R}^+$ *be any nonnegative and nondecreasing function. Then the weighted maximal operator* $\widetilde{\sigma}^{*,\nabla}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{\rho(n)(1/p-2)}\varphi(\rho(n))},$$

is bounded from the Hardy space H_p *to the Lebesgue space* L_p *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} < \infty. \tag{1}$$

For instance, if $\varphi(n) = n^{(1+\varepsilon)/p}$, then (1) is fulfilled for $\varepsilon > 0$, but not for $\varepsilon = 0$. Therefore

Main results

Corollary

a) *Let* $0 < p < 1/2$ *and* $f \in H_p(G)$. *Then the weighted maximal operator* $\widetilde{\sigma}^{*,\nabla,\varepsilon}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla,\varepsilon} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{\rho(n)(1/p-2)}(\rho(n))^{(1+\varepsilon)/p}}, \quad \varepsilon > 0,$$

is bounded from the Hardy space H_p *to the Lebesgue space* L_p.

b) *The weighted maximal operator* $\widetilde{\sigma}^{*,\nabla,0}$, *defined by*

$$\widetilde{\sigma}^{*,\nabla,0} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{2^{\rho(n)(1/p-2)}(\rho(n))^{1/p}},$$

is not bounded from the Hardy space H_p *to the Lebesgue space* L_p.

Thank you for your attention!