Two-dimensional Walsh-Nörlund means

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Visegrád, Hungary

Introduction

Let us denote by

$$\mathbb{Z}_2$$

the discrete cyclic group of order 2.
Let every subset be open.
Haar measure on $\mathbb{Z}_2$ is given in the way that the measure of a singleton is $1/2$.
The Walsh group is defined by

$$G := \mathop{\times}_{k=0}^{\infty} \mathbb{Z}_2.$$

The elements of G are of the form

$$x = (x_0, x_1, ..., x_k, ...)$$

with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition modulo 2,
the measure (denoted by μ) is the product measure and the topology is the product topology.
Fine's map $|.|: G \to [0,1[$ is defined by

$$|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1} \quad (x \in G).$$

Backwards,
$x \in [0,1[$ then it can be expressed in the number system based 2.

$$x := \sum_{j=0}^{\infty} x_j 2^{-j-1} \qquad x_j \in \{0,1\}.$$

Dyadic addition in the interval $[0,1[$:

$$x \dot{+} y := \sum_{j=0}^{\infty} (x_j + y_j \mod 2) 2^{-j-1}.$$

Introduction

Rademacher functions:

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Rademacher system is $(r_k : k \in \mathbb{N})$.
The Rademacher system is orthonormal, but it is not complete.

Introduction

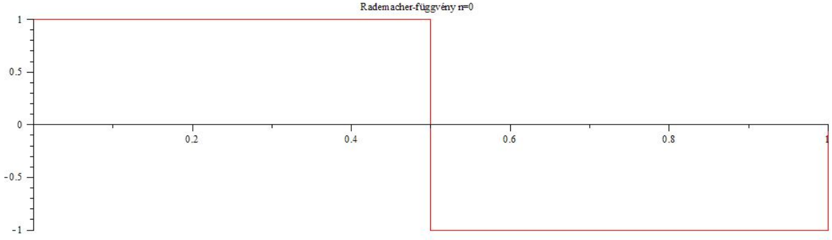

Figure: Rademacher function r_0

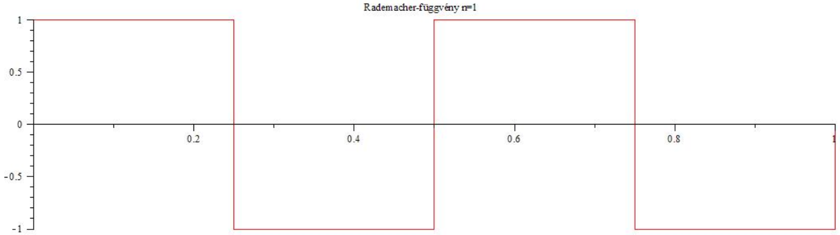

Figure: Rademacher function r_1

Introduction

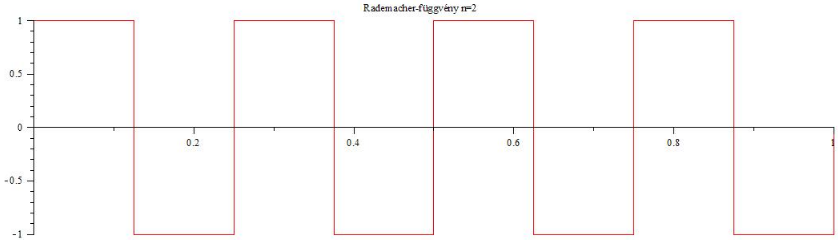

Figure: Rademacher function r_2

Introduction

If $n \in \mathbb{N}$, then

$$n = \sum_{i=0}^{\infty} \varepsilon_i(n) 2^i, \quad \varepsilon_i(n) \in \{0, 1\} \ (i \in \mathbb{N}).$$

Let the order of n be defined by

$$|n| := \max\{j \in \mathbb{N} : \varepsilon_j(n) \neq 0\}.$$

That is, $2^{|n|} \leq n < 2^{|n|+1}$.

Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{\varepsilon_k(n)} = (-1)^{\sum_{k=0}^{|n|} \varepsilon_k(n) x_k}$$

Walsh-Paley system: $(w_n : n \in \mathbb{N})$

Introduction

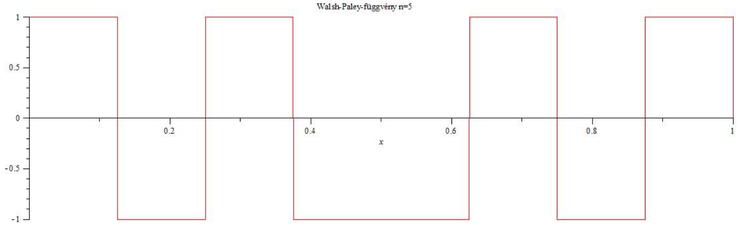

Figure: Walsh-Paley function w_5

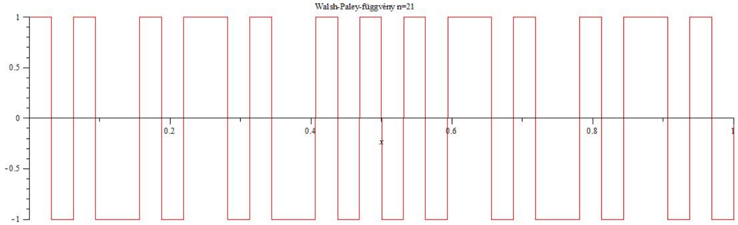

Figure: Walsh-Paley function w_{21}

$$w_{2^n} = r_n \quad (n \in \mathbb{N}).$$

Then

$$\{r_n : n \in \mathbb{N}\} \subset \{w_n : n \in \mathbb{N}\}.$$

The Walsh-Paley system is orthonormal and complete.
Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels:

$$\hat{f}(n) := \int_G f w_n,$$

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \quad D_n := \sum_{k=0}^{n-1} w_k,$$

$$\sigma_n(f) := \frac{1}{n} \sum_{k=0}^{n-1} S_k(f), \quad K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k,$$

$$S_n(f,x) = (f * D_n)(x), \quad \sigma_n(f,x) = (f * K_n)(x).$$

For Walsh-Paley-Fejér kernel functions we have $K_n^w(x) \to 0$, while $n \to \infty$ for every $x \neq 0$. However, it can take negative values.

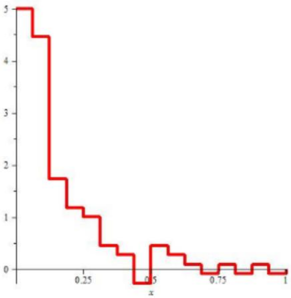

Figure: Fejér kernel K_{11}^w with respect to Walsh-Paley system

Nörlund means

Let us set $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers.
*n*th Nörlund mean of the Walsh-Fourier series

$$t_n(f;x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f;x),$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

It is always assumed that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. The summability method generated by the sequence $\{q_k : k \geq 0\}$ is regular if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

The Nörlund kernels are

$$F_n(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t).$$

The convolution

$$t_n(f;x) = (f * F_n)(x) = \int_G f(t)F_n(x+t)dt.$$

Examples:

- Fejér or arithmetic mean: $q_k = 1,\ 1 \le k \le n,\ Q_n = n$. The Fejér means has "good properties".
- The Nörlund's logarithmic method: the sequence $q_k = \frac{1}{k}$ is non-increasing (in sign $\downarrow$), $Q_n = l_n = \sum_{k=1}^{n-1} \frac{1}{k}$.

$$t_n(f;x) = \frac{1}{Q_n} \sum_{k=1}^{n-1} \frac{S_k(f;x)}{n-k}$$

Nörlund means

$(q_k,\ 1 \le k \le n)$ non-increasing (in sign $\downarrow$) sequences are:

- $(C,\alpha), \alpha \in (0,1)$ summability $q_k = A_k^{\alpha-1}$, $Q_n = A_n^\alpha$ where

$$A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!};$$

- Cesàro means with varying parameters $q_k = A_k^{\alpha_n-1}$, $Q_n = A_n^{\alpha_n}$, $\alpha_n \to 0$ as $n \to \infty$.

$(q_k,\ 1 \le k \le n)$ non-decreasing (in sign $\uparrow$) sequence

- $q_k = k^{\alpha-1}$, where $\alpha > 1$, $Q_n = n^\alpha$.

Nörlund kernel

Lemma (Kernel docomposition, Goginava and Nagy (2023))

Let $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$ *with* $n_1 > n_2 > \cdots > n_r \geq 0$. *Let us set* $n^{(0)} := n$ *and* $n^{(i)} := n^{(i-1)} - 2^{n_i}$ $(i = 1. \ldots, r)$. *Then the following decomposition holds.*

$$G_n^* = w_n F_n \;=\; \sum_{j=1}^{r} Q_{n^{(j-1)}} w_{2^{n_j}} D_{2^{n_j}} - \sum_{j=1}^{r} w_{n^{(j-1)}} w_{2^{n_j}-1} \sum_{k=1}^{2^{n_j}-1} q_{k+n^{(j)}} D_k .$$

Goginava, U. and Nagy, K., *Some Properties of The Walsh-Nörlund Means*, Quaestiones Mathematicae 46(2) (2023) 331-334.

Nörlund kernel

Lemma (Yano (1951), Toledo (2018))

$$\|K_n\|_1 \leq c \text{ for all } n \in \mathbb{N}.$$

$$c = 2.$$

S. Yano, On approximation by Walsh functions, Proc. Amer. Math. Soc. 2 (1951), 962-967.

$$c = \frac{17}{15}$$

R. Toledo, On the boundedness of the L_1-norm of Walsh-Fejér kernels, J. Math. Anal. Appl. 457(1) (2018), 153-178.

Nörlund kernel

Theorem (Goginava and Nagy (2023))

Let $\{q_k : k \in \mathbb{N}\}$ *be a sequence of non-negative numbers.*
a) If the sequence $\{q_k : k \in \mathbb{N}\}$ *is monotone non-decreasing (in sign* $q_k \uparrow$*). Then*

$$\|F_n\|_1 \leq \frac{17}{15} \quad \text{for all } n \in \mathbb{P}.$$

b) If the sequence $\{q_k : k \in \mathbb{N}\}$ *is monotone non-increasing (in sign* $q_k \downarrow$*). Then*

$$\|F_n\|_1 \sim \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k}.$$

Goginava, U. and Nagy, K., *Some Properties of The Walsh-Nörlund Means*, Quaestiones Mathematicae 46(2) (2023) 331-334.

Historical notes on Nörlund means

For monotone non-increasing sequence $\{q_k : k \in \mathbb{N}\}$ (in sign $q_k \downarrow$) we introduce the Nörlund variation of n by

$$V(n, \{q_k\}) := \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| Q_{2^k}.$$

- Móricz and Siddiqi (1992): The rate of the approximation by Nörlund means is estimated in terms of modulus of continuity in L_p spaces $(1 \leq p < \infty)$ and in C.

Móricz, F. and Siddiqi, A., Approximation by Nörlund means of Walsh-Fourier series, Journal of Approximation Theory 70, (1992), 375-389.

- Fridli, Manchada and Siddiqi (2008): The rate of the approximation by Nörlund means is estimated in terms of modulus of continuity in homogeneous Banach spaces and in dyadic Hardy spaces.

S. Fridli, P. Manchanda, and A.H. Siddiqi, Approximation by Walsh-Nörlund means, Acta Sci. Math. (Szeged) 74(3-4) (2008), 593-608.

Historical notes on Nörlund means

- Nagy (2012): The norm convergence of general two-dimensional Nörlund means is treated.

$$T_{n,m}(f;x,y)=\frac{1}{Q_{n,m}}\sum_{k=1}^{n-1}\sum_{l=1}^{m-1}q_{n-k,m-l}S_{k,l}(f;x,y), \quad Q_{n,m}=\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}q_{k,l}$$

Nagy, K., Approximation by Nörlund means of double Walsh-Fourier series for Lipschitz functions, Mathematical Inequalities and Applications 15 (2) (2012) 301-322.

- Goginava, Nagy (2023): Almost everywhere convergence of Nörlund means with monotone coefficients.

Goginava, U. and Nagy, K., *Some Properties of The Walsh-Nörlund Means*, Quaestiones Mathematicae 46(2) (2023) 331-334.

two-dimensional Nörlund means

For sequences $\{q_k : k \in \mathbb{N}\}$ and $\{p_l : l \in \mathbb{N}\}$ of non-negative numbers $(p_0, q_0 > 0)$, we define

$$t_{n,m}^{(q,p)}(f; x, y) := \frac{1}{Q_n P_m} \sum_{k=1}^{n} \sum_{l=1}^{m} q_{n-k} p_{m-l} S_{k,l}(f; x, y),$$

Kernel function:

$$F_{n,m}^{(q,p)}(x, y) = F_n^{(q)}(x) F_m^{(p)}(y).$$

Historical notes

- Móricz, Schipp and Wade (1992): the Fejér means

$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} S_{i,j}(f) \to f \text{ a.e.}$$

in Pringsheim sense (that is, $\min\{n,m\} \to \infty$) for all functions $f \in L \ln L\left(\mathbb{I}^2\right)$

Móricz F., Schipp F., Wade W.R. Cesàro summability of double Walsh-Fourier series. Trans. Amer. Math. Soc. 1992, 329 (1), 131-140.

- Gát (2000): the theorem of Móricz, Schipp and Wade can not be sharpened. For all measurable function $\delta : [0;+\infty) \to [0;+\infty)$; $\lim_{t\to\infty} \delta(t) = 0$, we have a function f such as $f \in L\log^+ L\delta(L)$ and $\sigma_n f$ does not converge to f a.e.

Gát G. On the divergence of the (C, 1) means of double Walsh-Fourier series. Proc. Amer. Math. Soc. 2000, 128 (6),1711-1720.

- Weisz (1996):

$$\frac{1}{A^{\alpha}_{n-1}A^{\beta}_{m-1}}\sum_{i=1}^{n}\sum_{j=1}^{m}A^{\alpha-1}_{n-i}A^{\beta-1}_{m-j}S_{ij}(f)\to f \text{ a.e.} \qquad (1)$$

as $\min\{n,m\}\to\infty$ $\alpha,\beta>0$, $f\in H_{\natural}(\mathbb{I}^2)$.

Weisz F. Cesàro summability of one-and two-dimensional Walsh-Fourier series. Anal. Math. 1996, 22 (3), 229-242.

- Simon (2000): Analogical result for $(C,1)$ means of Walsh-Kaczmarz system. General methods as well.

Simon P., Cesàro Summability with Respect to Two-Parameter Walsh Systems, Monatsh. Math. 131, 321-334 (2000).

Let $f \in L_1(\mathbb{I}^2)$, the hybrid maximal function is

$$f^{\natural}(x,y) := \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(t,y)dt \right|.$$

The space $H_{\natural}(\mathbb{I}^2)$ of Hardy type as the set of function f such that

$$\|f\|_{H^{\#}} := \left\|f^{\natural}\right\|_1 < \infty.$$

$f \in L_1(\mathbb{I}^2)$ belongs to the logarithmic space $L \ln L(\mathbb{I}^2)$ if the integral

$$\int_{\mathbb{I}^2} |f| \log^+ |f| < \infty.$$

$L \ln L(\mathbb{I}^2) \subset H_{\natural}(\mathbb{I}^2)$. $f \in L \ln L(\mathbb{I}^2)$ if and only if $|f| \in H^1_{\natural}(\mathbb{I}^2)$.

two-dimensional Nörlund means

Theorem (Móricz, Schipp and Wade (1992))

Let $\left(V_n^i, n \in \mathbb{N}\right), i = 0, 1,$ *be the sequence of* $L_1(\mathbb{I})$ *functions. Define one-dimensional operators*

$$T^i f := \sup_{m\in\mathbb{N}} \left|f * V_m^i\right|, \widetilde{T}^i f := \sup_{m\in\mathbb{N}} \left|f * \left|V_m^i\right|\right|$$

for $f \in L_1(\mathbb{I}), i = 0, 1,$ *and suppose that there exist absolute constants* c_0, c_1 *such that*

$$\mu\left(\left\{\widetilde{T}^0 f > \lambda\right\}\right) \leq \frac{c_0}{\lambda}\|f\|_1, \ \ and \ \ \left\|T^1 f\right\|_1 \leq c_1 \|f\|_{H_1}$$

for $f \in L_1(\mathbb{I})$ *and* $\lambda > 0$*. If* $Tf := \sup_{(n,m)\in\mathbb{N}^2} \left|f * \left(V_n^0 \otimes V_m^1\right)\right|$*, then*

$$\mu(\{Tf > \lambda\}) \leq \frac{c_0 c_1}{\lambda}\|f\|_{H_\natural} \quad \left(f \in H_\natural\left(\mathbb{I}^2\right), \lambda > 0\right).$$

For $K > 0$ $\qquad L_K(\{q_k\}) :=$

$$:= \left\{ n \in \mathbb{N} : V(n, \{q_k\}) := \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_{k+1}(n) - \varepsilon_k(n)| \, Q_{2^k} \le K \right\}.$$

Let us set $\widetilde{t}_{m_A}^{(q)}(f) := f * \left| F_{m_A}^{(q)} \right|$.

Theorem (Goginava, Nagy (2023))

Let $\{m_A : A \in \mathbb{P}\}$ be a strictly monotone increasing sequence. Let $\{q_k : k \in \mathbb{N}\}$ be a monotone non-increasing sequence of non-negative numbers (in sign $q_k \downarrow$). If

$$\{m_A : A \in \mathbb{N}\} \subset L_K(\{q_k\})$$

for some $K > 0$. Then there exists a positive constant c such that

$$\sup_{\lambda > 0} \lambda \mu \left(\{ \sup_A |\widetilde{t}_{m_A}^{(q)}(f)| > \lambda \} \right) \le c \|f\|_1 \quad (f \in L_1(\mathbb{I}), \lambda > 0).$$

Theorem (Goginava (2023))

Let $\{q_k\}$ be non-increasing positive sequence (in sign $q_k \downarrow$). $t^*(f) = \sup_n |t_n(f)|$ is bounded from the Hardy space $H_1(\mathbb{I})$ to the space $L_1(\mathbb{I})$ if and only if

$$\sup_N \frac{1}{Q_{2^N}} \sum_{j=1}^{N} Q_{2^j} < \infty.$$

Goginava, U., *Maximal operator of Walsh-Nörlund means on the dyadic Hardy spaces, Acta Math. Hungar. 169 (2023) 171-190.*

Lemma (Goginava, Nagy (2024))

Let $\{q_l : \ l \in \mathbb{N}\}$ be a monotone non-decreasing sequence of non-negative numbers ($q_l \uparrow$). Then for the operator $\widetilde{t}(f) := \sup_{n\in\mathbb{N}} |f * |F_n||$ is of weak type (1,1).

U. Goginava and K. Nagy, *Almost everywhere convergence of two-dimensional Walsh-Nörlund means*, Carpathian Mathematical Publications 16(1) (2024) 290-302.

Let $\{q_k : \ k \in \mathbb{N}\}$ be a monotone non-decreasing sequence of non-negative numbers ($q_l \uparrow$). It follows that

$$t^*(f) \leq c \sup_n |\sigma_n(f)| = c\sigma^*(f)$$

Fujii proved that $\sigma^*(f)$ is bouded from H_1 to L_1.

Theorem (Goginava, Nagy (2024))

Let $\{p_k : k \in \mathbb{N}\}$, $\{q_k : k \in \mathbb{N}\}$ *are non-increasing seq.* $(q_k, p_k \downarrow)$,

$$\{n_A : A \in \mathbb{N}\} \subset L_K(\{q_k\}) \quad \text{for some } K > 0 \text{ and}$$

$$\sup_m \left(\frac{1}{P_{2^m}} \sum_{k=1}^{m} P_{2^k} \right) < \infty.$$

Then the maximal operator $\sup_{A,m\in\mathbb{N}} \left| f * F_{n_A}^{(q)} \otimes F_m^{(p)} \right|$ *is boundend from the space* $H_\natural\left(\mathbb{I}^2\right)$ *to the space weak* $- L_1\left(\mathbb{I}^2\right)$.

Corollary

Let the conditions of previous Theorem be satisfied. Then $t_{n_A,m}(f)$ *converge to* f *a.e. as* $\min\{n_A, m\} \to \infty$ *for all* $f \in H_\natural(\mathbb{I}^2)$.

two-dimensional Nörlund means

Theorem (Goginava, Nagy (2024))

Let $\{q_k : k \in \mathbb{N}\}$ *be non-increasing sequence* $(q_k \downarrow)$ *such that,*

$$\{n_A : A \in \mathbb{N}\} \subset L_K\left(\{q_k\}\right) \quad \text{for some } K > 0$$

and let $\{p_k : k \in \mathbb{N}\}$ *be increasing (positive) sequence* $(p_k \uparrow)$*.*
Then the maximal operator
$\sup\limits_{A,m\in\mathbb{N}} \left| f * F_{n_A}^{(q)} \otimes F_m^{(p)} \right|$ *is boundend from the space* $H_\natural\left(\mathbb{I}^2\right)$ *to the space weak* $- L_1\left(\mathbb{I}^2\right)$*.*

Corollary

Let the conditions of previous Theorem be satisfied. Then $t_{n_A,m}(f) \to f$ *a.e. as* $\min\{n_A, m\} \to \infty$ *for all* $f \in H_\natural(\mathbb{I}^2)$*.*

Theorem (Goginava, Nagy (2024))

Let $\{q_k : \ k \in \mathbb{N}\}$ and $\{p_l : \ l \in \mathbb{N}\}$ be monotone non-decreasing sequences of non-negative numbers ($q_k, p_k \uparrow$). Then there exists a positive constant c such that

$$|\{\sup_{n,m} |t_{n,m}(f)| > \lambda\}| \leq \frac{c}{\lambda} \|f^{\natural}\|_1$$

holds for all $f \in H^1_{\natural}(\mathbb{I}^2)$.

Corollary

Let the conditions of previous Theorem be satisfied. Then the two-dimensional Walsh-Nörlund means $t_{n,m}(f) \to f$ almost everywhere as $\min\{n, m\} \to \infty$ for all functions $f \in H_{\natural}(\mathbb{I}^2)$.

Thanks for your attention!