Approximation by matrix transform means with respect to the Walsh system in Lebesgue spaces

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Happy Birthday to

Ferenc Schipp (85), Péter Simon (75), László Szili (70), Ferenc Weisz (60)

and the Department of Numarical Analysis (40)!

18 Oct, 2024

General introduction

Denote by $\mathbb{P}$ the set of the positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$.

Dyadic group

Denote by $\mathbb{Z}_2 := \{0, 1\}$ the additive group of integers modulo 2. Define the dyadic group G as the complete direct product of the groups $\mathbb{Z}_2$ with the product of the discrete topologies of $\mathbb{Z}_2$'s. The group operation is the modulo 2 addition.

The elements of G are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots), \text{ where } x_n \in \mathbb{Z}_2 \text{ and } n \in \mathbb{N}.$$

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Dyadic intervals

$$I_0(x) := G,$$

$$I_n(x) := \{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \; (x \in G, n \in \mathbb{P}).$$

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Measure

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/2, \text{ where } j \in \mathbb{Z}_2$$

is the Haar measure on G with $\mu(G) = 1$.

L_p spaces

Let $L_p(G)$ denote the usual Lebesgue spaces on G with corresponding norms $\|.\|_p$, where $1 \leq p < \infty$ and $C(G)$ denote the space of continuous functions on G with the norm $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

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$L_p(G)$ modulus of continuity

$$\omega_p(f,\delta) := \sup_{|t|<\delta} \|f(.+t) - f(.)\|_p ,$$

for $f \in L_p(G)$, where $\delta > 0$ with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

In the case $f \in C(G)$ we change p by ∞.

Walsh-Paley system

Rademacher functions

$$r_k(x) := (-1)^{x_k},$$

so $r_k(x) : G \to \{-1, 1\}$, where $x \in G, \; k \in \mathbb{N}$

Walsh-Paley system

$w := \{w_n : n \in \mathbb{N}\}$ on G

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \;\; (n \in \mathbb{N}).$$

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Some of the usual definitions of the Walsh-Fourier analysis.

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Walsh-Fejér mean

$$\sigma_n(f) := \frac{1}{n}\sum_{k=1}^{n} S_k(f)$$

Walsh-Dirichlet kernels

$$D_n := \sum_{k=0}^{n-1} w_k,$$

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Nörlund mean of the Walsh-Fourier series

Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-negative numbers.

$$t_n(f;x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f;x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \in \mathbb{P})$, $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$.

Matrix transform

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a infinite upper triangular matrix of numbers.

Matrix transform mean

$$\sigma_n^T(f;x) := \sum_{k=1}^{n} t_{k,n} S_k(f;x),$$

$\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{P}\}$ $(n \in \mathbb{P})$.

Matrix transform kernel

$$K_n^T(x) := \sum_{k=1}^{n} t_{k,n} D_k(x).$$

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Historical overview

In classical book of Schipp, Wade, Simon and Pál [7], on p. 191. we can read inequality

$$\|\sigma_{2^s}(f) - f\|_X \leq \omega^{(X)}\left(f; 2^{-s}\right) + \sum_{k=0}^{s-1} 2^{k-s} \omega^{(X)}\left(f; 2^{-k}\right),$$

where X is a homogeneous Banach space (for example any L^p space, where $1 \leq p < \infty$ and the space of continuous functions C).

Historical overview

Móricz and Siddiqi [5] (1992)

Let $f \in L_p(G)$, $1 \leq p \leq \infty$ and let $\{q_k : k \in \mathbb{N}\}$ be a sequence of nonnegative numbers such that $\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1)$ for some $1 < \gamma \leq 2$.

a) If q_k is non-decreasing, then

$$\|t_n(f) - f\|_p \leq \frac{5}{2Q_n} \sum_{j=0}^{|n|-1} 2^j q_{n-2^j} \omega_p \left(f, \frac{1}{2^j}\right) + c\omega_p \left(f, \frac{1}{2^{|n|}}\right).$$

b) If q_k is non-increasing, then

$$\|t_n(f) - f\|_p \leq \frac{5}{2Q_n} \sum_{j=0}^{|n|-1} (Q_{n-2^j-1} - Q_{n-2^{j+1}-1}) \omega_p \left(f, \frac{1}{2^j}\right) + c\omega_p \left(f, \frac{1}{2^{|n|}}\right).$$

Historical overview

Blahota and K. Nagy [3] (2018)

Let $f \in L_p(G), 1 \leq p \leq \infty$. For every $n \in \mathbb{N}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that $\sum_{k=1}^{n} t_{k,n} = 1$ is satisfied.

a) If the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-decreasing for a fixed n and the condition $t_{n,n} = O\left(\frac{1}{n}\right)$ is satisfied, then

$$\left\|\sigma_n^T(f) - f\right\|_p \leq 5 \sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \omega_p\left(f, \frac{1}{2^j}\right) + c\omega_p\left(f, \frac{1}{2^{|n|}}\right).$$

b) If the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ is non-increasing for a fixed n, then

$$\left\|\sigma_n^T(f) - f\right\|_p \leq 5 \sum_{j=0}^{|n|-1} 2^j t_{2^j,n} \omega_p\left(f, \frac{1}{2^j}\right) + c\omega_p\left(f, \frac{1}{2^{|n|}}\right).$$

Historical overview - motivation

Areshidze and Tephnadze [1] (2024)

Let $f \in L_p(G),\ 1 \leq p < \infty$ and let t_n be a regular Nörlund mean generated by non-decreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then

$$\|t_n(f) - f\|_p \leq 18 \sum_{j=0}^{|n|-1} 2^j \frac{q_{n-2^j}}{Q_n} \omega_p\left(f, \frac{1}{2^j}\right) + 12\omega_p\left(f, \frac{1}{2^{|n|}}\right).$$

Auxiliary results

Paley's lemma

$$D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n(0), \\ 2^n, & \text{if } x \in I_n(0). \end{cases}$$

The improved version of Yano's lemma:

Toledo [8] (2018)

$$\sup_{n \in \mathbb{P}} \|K_n\|_1 = \frac{17}{15}.$$

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Auxiliary results

Gát [4] (1998)

Let $n, t \in \mathbb{N}$ and $t < n$. Then

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_t(0) \backslash I_{t+1}(0),\ x - e_t \in I_n(0), \\ \frac{2^n+1}{2}, & \text{if } x \in I_n(0), \\ 0, & \text{otherwise.} \end{cases}$$

Persson, Tephnadze and Weisz [6] (2022)

Let $n \in \mathbb{N}$ and $f \in L_p(G)$ for some $1 \leq p < \infty$. Then we have inequality

$$\|\sigma_n(f) - f\|_p \leq 3 \sum_{s=0}^{|n|} \frac{2^s}{2^{|n|}} \omega_p\left(f, \frac{1}{2^s}\right).$$

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New results

Blahota and D. Nagy [2] (2024)

Let $f \in L_p(G), 1 \le p \le \infty$. For every $n \in \mathbb{N}$, $\{t_{k,n} : 1 \le k \le n\}$ be a finite sequence of non-negative numbers such that

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied. If the finite sequence $\{t_{k,n} : 1 \le k \le n\}$ is non-increasing for a fixed n, then we have

$$\left\|\sigma_n^T(f) - f\right\|_p \le \frac{31}{15} \sum_{k=0}^{|n|-1} 2^k t_{2^k,n} \omega_p\left(f, \frac{1}{2^j}\right) + \frac{47}{30} \omega_p\left(f, \frac{1}{2^{|n|}}\right).$$

New results

Blahota and D. Nagy [2] (2024)

Let the finite sequence $\{t_{k,2^n} : 1 \leq k \leq 2^n\}$ of non-negative numbers be non-decreasing for all $n \in \mathbb{N}$ and

$$\sum_{k=1}^{2^n} t_{k,2^n} = 1.$$

Then for any $f \in L_p(G)$ for some $1 \leq p < \infty$, we have the following inequality

$$\left\|\sigma_{2^n}^T(f) - f\right\|_p \leq \sum_{s=0}^{n-1} \frac{2^s}{2^n} \omega_p\left(f, \frac{1}{2^s}\right) + 3\sum_{s=0}^{n-1}(n-s)2^s t_{2^n-2^s+1,2^n} \omega_p\left(f, \frac{1}{2^s}\right) + \left(2 + \frac{1}{2^n}\right)\omega_p\left(f, \frac{1}{2^n}\right).$$

New results

Blahota and D. Nagy [2] (2024)

For every $n \in \mathbb{P}$, let the finite sequence $\{t_{k,n} : 1 \leq k \leq n\}$ of non-negative numbers be non-decreasing for all n and we suppose that

$$\sum_{k=1}^{n} t_{k,n} = 1 \quad \text{and} \quad t_{n,n} = O\left(\frac{1}{n}\right).$$

Then for any $f \in L_p(G)$ for some $1 \leq p < \infty$, we have the following inequality

$$\left\|\sigma_n^T(f) - f\right\|_p \leq c \sum_{k=0}^{|n|} \frac{2^k}{2^{|n|}} \omega_p\left(\frac{1}{2^k}, f\right).$$

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Thank you for your attention!