Hardy-Littlewood típusú egyenlőtlenségek Fourier-transzformáltakra

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Lebesgue and Lorentz spaces

The $L_p(\mathbb{R}^d)$ space is equipped with the quasi-norm

$$\|f\|_{L_p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p} \qquad (0 < p < \infty).$$

The non-increasing rearrangement of a measurable function f is given by

$$f^*(t) := \inf \{ \rho : |\{|f| > \rho\}| \le t \} \qquad (t > 0).$$

For a multi-dimensional measurable function and for fixed variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_d$, by $f^{*_i}(y_1, \ldots, y_{i-1}, \cdot, y_{i+1}, \ldots, y_d)$, we denote the non-increasing rearrangement with respect to the i-th variable $(i = 1, \ldots, n)$. Let $(j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. Applying the non-increasing rearrangement in all variables consecutively, we obtain

$$f^{*_{j_1}, \ldots, *_{j_d}} := ((f^{*_{j_1}})^{*_{j_2}} \ldots)^{*_{j_d}} .$$

Let $\mathbf{p} = (p_1, \ldots, p_d)$ and $\mathbf{q} = (q_1, \ldots, q_d)$ with $0 < p_j < \infty$ and $0 < q_j \le \infty$, $j = 1, 2 \ldots n$. Let $(j_1, j_2, ..., j_d)$ be a permutation of $(1, 2, ..., d)$. In this case we will write $0 < \mathbf{p} < \infty$ and $0 < \mathbf{q} \le \infty$. The Lorentz spaces $L_{p,q}(\mathbb{R}^d)$ and $L^*_{\mathbf{p},\mathbf{q}}(\mathbb{R}^d)$ consist of all measurable functions f for which

$$\|f\|_{L_{p,q}} := \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

and

$$\|f\|_{L^*_{\mathbf{p},\mathbf{q}}}$$

$$:= \left(\int_0^\infty \ldots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} \ldots t_2^{\frac{1}{p_d}} f^{*_{j_1} \cdots *_{j_d}}(t_1, \ldots, t_d) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \ldots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}}$$

are finite.

Note that $L_{p,p} = L_p$, however, the space $L^*_{\mathbf{p},\mathbf{q}}$ with $\mathbf{p} = \mathbf{q}$ does not coincide with the mixed Lebesgue space $(L_{p_1}, ..., L_{p_d})$. However, if $p_i = q_i = p$, $i = 1, 2, ..., n$, then $L^*_{\mathbf{p},\mathbf{q}} = L_p$.

Fourier transform

For $f \in L_1(\mathbb{R}^d)$, the Fourier transform is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i(\xi,x)} dx.$$

We can extend this definition to tempered distributions in the usual way. Let $Q_N := [-N, N]^d$. For $f \in L_p(\mathbb{R}^d)$, we define

$$(\mathfrak{F}_N f)(\xi) := \int_{Q_N} f(x) e^{-i(\xi,x)} dx.$$

As it is well known, if $f \in L_p(\mathbb{R}^d)$, $1 \leq p \leq 2$, then

$$\widehat{f} = \lim_{N \to +\infty} (\mathfrak{F}_N f) \qquad \text{in the } L_{p'}(\mathbb{R}^d)\text{-norm}.$$

The Fourier transform of $f \in L_p(\mathbb{R}^d)$, $2 < p < \infty$, is not well-defined as a function.

Pitt's inequality

Hausdorff–Young inequality states that

$$\left\|\widehat{f}\right\|_{p'} \lesssim \|f\|_p, \qquad 1 < p \le 2, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

A counterpart of this inequality is the following Hardy-Littlewood inequality

$$\left(\int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p \, dt\right)^{1/p} \lesssim \|f\|_p, \qquad 1 < p \le 2, \qquad (1)$$

where $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. In particular, (1) sharpens the well-known inequality

$$\left(\int_{\mathbb{R}^d} |t|^{d(p-2)} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_p, \qquad 1 < p \le 2, \qquad (2)$$

where $|t|$ denotes the Euclidean norm of t.

For $1 < r \le q < \infty$ and $f \in \bigcup_{1 \le s \le 2} L_s$,

$$\left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^\beta \left| \widehat{f}(t) \right| \right)^q dt \right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d} \left((|x_1| \dots |x_d|)^\alpha |f(x)| \right)^r dx \right)^{1/r} \tag{3}$$

provided that

$$0 \le \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}$$

and, additionally,

$$\beta \le 0,$$

which is equivalent to $\frac{1}{r'} - \frac{1}{q} \le \alpha$.

Now we point out three well-known special cases. For $\alpha = \beta = 0$, $q = r'$, $1 < r \leq 2$,

$$\left\|\widehat{f}\right\|_{r'} \lesssim \|f\|_r$$

which is the Hausdorff-Young inequality;
for $\alpha = 0$, $\beta = 1 - 2/r$, $1 < r = q \leq 2$, we obtain (1):

$$\left(\int_{\mathbb{R}^d} (|t_1| \ldots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p \, dt\right)^{1/p} \lesssim \|f\|_p \,, \qquad 1 < p \leq 2,$$

and for $\beta = 0$, $\alpha = 1 - 2/r$, $2 \leq r = q < \infty$, we establish the dual to (1), that is,

$$\left\|\widehat{f}\right\|_r \lesssim \left(\int_{\mathbb{R}^d} (|x_1| \ldots |x_d|)^{r-2} |f(x)|^r \, dx\right)^{1/r}.$$

Pitt's inequality for Hardy-Cesàro and Hardy-Bellman operators

For a one-dimensional function, the Hardy-Cesàro and Hardy-Bellman operators are defined by

$$\mathcal{H}f(t) := \frac{1}{t}\int_0^t f(x)\,dx, \qquad \mathcal{B}f(t) := \int_{|t|}^{\infty} f(\mathrm{sign}\,(t)x)\,\frac{dx}{x}.$$

Let $E := \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) : \varepsilon_j = 0, 1\}$. In the multi-dimensional case set $\mathcal{H}_\varepsilon := \mathcal{H}_{\varepsilon_d,d} \ldots \mathcal{H}_{\varepsilon_2,2}\mathcal{H}_{\varepsilon_1,1}$, where $\varepsilon \in E$ and

$$(\mathcal{H}_{\varepsilon_i,i}f)(t) := \begin{cases} \frac{1}{t_i}\int_0^{t_i} f(x_1, \ldots, x_i, \ldots, x_d)\,dx_i, & \text{if} \quad \varepsilon_i = 0 \\ \int_{|t_i|}^{\infty} f(x_1, \ldots, \mathrm{sign}\,(t_i)x_i, \ldots, x_d)\,\frac{dx_i}{x_i}, & \text{if} \quad \varepsilon_i = 1. \end{cases}$$

Theorem

Suppose that $f \in \bigcup_{1 \le s \le 2} L_s$. *Let* $1 < r \le q < \infty$ *and*

$$0 \le \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}$$

and $(j_1, j_2, \ldots, j_d)$ *be a permutation of* $(1, 2, \ldots, d)$. *Then, for any* $\varepsilon \in E$,

$$\left(\int_{\mathbb{R}^d} \left((|t_1| \ldots |t_d|)^{\beta} \left| \mathcal{H}_{\varepsilon} \widehat{f}(t) \right| \right)^q dt \right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d} \left((|x_1| \ldots |x_d|)^{\alpha} \left| f^{*_{j_1} \cdots *_{j_d}}(x) \right| \right)^r dx \right)^{1/r}. \tag{4}$$

Note that the right hand side of (4) is $\|f\|_{L^*_{p,r}}$ with $1/p = \alpha + 1/r$.

Theorem

Under the same conditions

$$\left(\int_{\mathbb{R}^d}\left((|t_1|\dots|t_d|)^{\beta}\left|\mathcal{H}_\varepsilon\widehat{f}(t)\right|\right)^q dt\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d}\left((|x_1|\dots|x_d|)^{\alpha}|f(x)|\right)^r dx\right)^{1/r}. \tag{5}$$

In particular, if $\alpha = 0$, $\beta = 1 - 2/r$, $1 < r = q < \infty$, then

$$\left(\int_{\mathbb{R}^d}(|t_1|\dots|t_d|)^{r-2}\left|\mathcal{H}_\varepsilon\widehat{f}(t)\right|^r dt\right)^{1/r} \lesssim \|f\|_r.$$

Theorem

Suppose that $f \in \bigcup_{1 \le s \le 2} L_s$. *Let* $1 < r \le q < \infty$ *and*

$$\frac{1}{r'} - \frac{1}{q} \le \alpha = \frac{1}{r'} - \frac{1}{q} - \beta < \frac{1}{r'}.$$

Then, for any $\varepsilon \in E$,

$$\left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^{\beta} \left| \mathcal{H}_\varepsilon \widehat{f}(t) \right| \right)^q dt \right)^{1/q} \lesssim \left(\int_{\mathbb{R}^d} \left((|x_1| \dots |x_d|)^{\alpha} |f(x)| \right)^r dx \right)^{1/r}.$$

In particular, if $\beta = 0$, $\alpha = 1 - 2/r$ and $1 < r = q < \infty$, then

$$\left\| \mathcal{H}_\varepsilon \widehat{f} \right\|_r \lesssim \left(\int_{\mathbb{R}^d} (|x_1| \dots |x_d|)^{r-2} \, |f(x)|^r \, dx \right)^{1/r}.$$

Remark

If additionally, $\beta \le 0$, then (5) is weaker than (3), i.e.,

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^\beta \left| \mathcal{H}_\varepsilon \widehat{f}(t) \right| \right)^q dt \right)^{1/q} \\ &\quad \lesssim \left(\int_{\mathbb{R}^d} \left((|t_1| \dots |t_d|)^\beta \left| \widehat{f}(t) \right| \right)^q dt \right)^{1/q} \\ &\quad \lesssim \left(\int_{\mathbb{R}^d} \left((|x_1| \dots |x_d|)^\alpha \, |f(x)| \right)^r dx \right)^{1/r}. \end{aligned}$$

Hausdorff–Young inequality and net spaces

Let us denote by M the collection of all rectangles $I = I_1 \times ... \times I_d$ with sides parallel to the axes. We define the average function by

$$\overline{f}(t_1, ..., t_d) := \sup_{I \in M, |I_i| \geq t_i} \frac{1}{|I_1| \dots |I_d|} \left| \int_I f(x)\, dx \right|.$$

f belongs to the net space $\mathcal{N}_{\mathbf{p},\mathbf{q}}(M)$ $(0 < \mathbf{p}, \mathbf{q} \leq \infty)$ if

$$\|f\|_{\mathcal{N}_{\mathbf{p},\mathbf{q}}} = \left(\int_0^\infty \dots \left(\int_0^\infty \left(t_1^{\frac{1}{p_1}} \dots t_d^{\frac{1}{p_d}} \overline{f}(t_1, \dots, t_d) \right)^{q_1} \frac{dt_1}{t_1} \right)^{\frac{q_2}{q_1}} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q_d}}$$

is finite. $\mathcal{N}_{\mathbf{p},\mathbf{q}}$ is a normed linear space.
For $p_i = p,\ q_i = q,\ i = 1, 2, ..., n$, we also use the notation

$$\|f\|_{\mathcal{N}_{p,q}} = \left(\int_0^\infty \dots \int_0^\infty \left((t_1 \dots t_d)^{\frac{1}{p}} \overline{f}(t_1, \dots, t_d) \right)^q \frac{dt_1}{t_1} \dots \frac{dt_d}{t_d} \right)^{\frac{1}{q}}.$$

Recall that the Hardy–Littlewood–Paley estimate for classical Lorentz spaces states that

$$\left\|\widehat{f}\right\|_{L_{p',q}} \lesssim \|f\|_{L_{p,q}}, \quad 1 < p < 2, \quad 0 < q \leq \infty.$$

Theorem

Suppose that $f \in \bigcup_{1\leq s\leq 2} L_s$. *Let* $* = (j_1, j_2, ..., j_d)$ *be a permutation of* $(1, 2, ..., d)$. *If* $1 < \mathbf{p} < \infty$, $0 < \mathbf{q} \leq \infty$ *and* $f \in L^*_{\mathbf{p},\mathbf{q}}$, *then*

$$\left\|\widehat{f}\right\|_{\mathcal{N}_{\mathbf{p}',\mathbf{q}}} \lesssim \|f\|_{L^*_{\mathbf{p},\mathbf{q}}}.$$

In particular, for $1 < p < \infty$,

$$\left(\int_0^\infty \dots \int_0^\infty (t_1 \dots t_d)^{p-2} \left|\overline{\widehat{f}}(t_1, ..., t_d)\right|^p \, dt\right)^{1/p} \lesssim \|f\|_{L_p}.$$

Product Hardy spaces

We consider the so called product Hardy space $H_p = H_p(\mathbb{R} \times \cdots \times \mathbb{R})$, that is different from $H_p(\mathbb{R}^n)$.
For $\phi \in S(\mathbb{R})$ with $\int_{\mathbb{R}} \phi \, d\lambda \neq 0$, let

$$\phi_t(x) := t^{-1}\phi(x/t) \qquad (t > 0).$$

Then we say that a tempered distribution f is in the product Hardy space $H_p = H_p(\mathbb{R} \times \cdots \times \mathbb{R})$ $(0 < p < \infty)$ if

$$\|f\|_{H_p} := \left\| \sup_{t_1>0,\ldots,t_d>0} \left|(f * (\phi_{t_1} \otimes \cdots \otimes \phi_{t_d})\right| \right\|_p < \infty,$$

where $*$ denotes the convolution and

$$(\phi_{t_1} \otimes \cdots \otimes \phi_{t_d})(x) := \prod_{j=1}^{d} \phi_{t_j}(x_j), \qquad x \in \mathbb{R}^d.$$

It is known that different Schwartz functions yield equivalent norms.
Moreover, H_p is equivalent to L_p for $1 < p < \infty$.

Inequalities in Hardy spaces

The generalization of (2) to Hardy spaces is known:

$$\left(\int_{\mathbb{R}^d} |t|^{d(p-2)} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_{H_p(\mathbb{R}^d)}, \qquad 0 < p \leq 1.$$

The case $p = 1$ is called Hardy's inequality. We generalize (1) to Hardy spaces.

Theorem

If $0 < p \leq 1$ *and* $f \in H_p$*, then*

$$\left(\int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_{H_p}. \tag{6}$$

Since $\widehat{f}$ is a locally integrable function if $f \in H_p$ with $0 < p \leq 1$, the integral on the left hand side is well defined.

The Hardy–Cesàro operator is defined by

$$\mathcal{H}f := \mathcal{H}_0 f = \frac{1}{t_1 \cdots t_d} \int_0^{t_1} \ldots \int_0^{t_d} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d .$$

Theorem

If $0 < p \leq 1$ and $f \in H_p \bigcap \bigcup_{1 \leq q \leq 2} L_q$, then

$$\left(\int_{\mathbb{R}^d} (|t_1| \cdots |t_d|)^{p-2} \left| \widehat{\mathcal{H}f}(t) \right|^p \, dt \right)^{1/p} \lesssim \|f\|_{H_p} . \tag{7}$$

Remark

(i) For $p = 1$, (7) holds for all $f \in H_1$.
(ii) The left-hand sides of (6) and (7) are not comparable in general.

Let us recall the classical Hardy's inequalities. First,

$$\|\mathcal{H}_\varepsilon f\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

If all $\varepsilon_i = 0$ or all $\varepsilon_i = 1$, then even more is true. We just introduced the Hardy–Cesàro operator $\mathcal{H}f$ and now we define the Hardy–Bellman operator by

$$\mathcal{B}f := \mathcal{H}_1 f = \int_{|t_1|}^{\infty} \dots \int_{|t_d|}^{\infty} f(\operatorname{sign}(t_1)x_1, \dots, \operatorname{sign}(t_d)x_d) \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d}.$$

Then

$$\|\mathcal{H}f\|_p \lesssim \|f\|_p \quad (1 < p \leq \infty) \quad \text{and} \quad \|\mathcal{B}f\|_p \lesssim \|f\|_p \quad (1 \leq p < \infty). \tag{8}$$

Remark

If $f, \widehat{f} \in \bigcup_{1\le s\le 2} L_s$ *(say* $f \in \mathcal{S}$*) and* $1 < p \le 2$*, then*

$$\|\mathcal{H}_\varepsilon f\|_p \lesssim \left(\int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt\right)^{1/p} \lesssim \|f\|_p.$$

In particular, if $f \ge 0$ *is non-increasing and even in each direction, then*

$$\|f\|_p \asymp \left(\int_{\mathbb{R}^d} (|t_1| \dots |t_d|)^{p-2} \left|\widehat{f}(t)\right|^p dt\right)^{1/p}.$$

Equivalences of this type are usually called Hardy–Littlewood theorems or Boas-type results.

Even though $\mathcal{H}$ is not bounded in H_p, we derive the following weaker result, which can be considered as a generalization of inequality (8).

Theorem

If $0 < p \leq 1$ *and* $f \in H_p \bigcap \bigcup_{1<q\leq 2} L_q$*, then*

$$\|\mathcal{H}f\|_p \lesssim \|f\|_{H_p}.$$

For the operator $\mathcal{B}$ the situation is symmetric. It is easy to see that $\mathcal{B}$ is trivially bounded from H_1 to L_1 since

$$\|\mathcal{B}f\|_1 \leq C\|f\|_1 \leq C\|f\|_{H_1}.$$

In fact, it turns out that the operator $\mathcal{B}$ is bounded in H_p, i.e.,

Theorem

If $0 < p \leq 1$ *and ff* $\in H_p \bigcap \bigcup_{1\leq q\leq 2} L_q$*, then*

$$\|\mathcal{B}f\|_{H_p} \lesssim \|f\|_{H_p}.$$